Asymptotic Behavior of Certain Branching Processes

by

Erin M. Beckman

Department of Mathematics
Duke University

Date: ______________________

Approved:

____
James Nolen, Supervisor

____
Jonathan Mattingly

____
Richard Durrett

____
Michael Reed

Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Mathematics
in the Graduate School of Duke University
2019

Abstract

Asymptotic Behavior of Certain Branching Processes

by

Erin M. Beckman

Department of Mathematics
Duke University

Date: ____________________
Approved:

James Nolen, Supervisor

Jonathan Mattingly

Richard Durrett

Michael Reed

An abstract of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the Graduate School of Duke University
2019

Abstract

This dissertation examines the asymptotic behavior of three branching processes. The first is a branching process with selection; the selection is dictated by a fitness function which is the sum of a linear part and a periodic part. It is shown that the system has an asymptotic speed and that there is a stationary distribution in an appropriate moving frame. This is done through an examination of tightness of the process and application of an ergodic theorem. The second process studied is a branching process with selection driven by a symmetric function with a single local maximum at the origin and which monotonically decreases away from the origin. For this process, a large particle limit of the system is proven and related to the solution to a free boundary partial differential equation. Finally, a branching process is studied in which the branch rate of particles is a function of the empirical measure. Weak convergence to the solution of a non-local partial differential equation is proven. Tightness is proven first, and then the limit object is characterized by its behavior when applied to test functions.

Dedicated to my husband.

Contents

List of Figures

List of Abbreviations and Symbols

Symbols

$\mathbb{P}$	probability measure
$\mathcal{M}_+(Y)$	the space of positive, finite measures on Y
$\mathcal{M}_1(Y)$	the space of probability measures Y
$D([0,T],Y)$	the Skorohod space of functions from $[0,T]$ to Y
$C_0(\mathbb{R},\mathbb{R})$	the set of continuous functions which go to 0 as x goes to infinity and negative infinity
$C_c^\infty(\cdot,\mathbb{R})$	the set of smooth, compactly supported functions
$C_b^\infty(\cdot,\mathbb{R})$	the set of smooth, bounded functions
$\stackrel{d}{=}$	equivalence in distribution
$\mathrm{Geo}(p)$	a geometric distribution with parameter p
$\mathrm{Poi}(\lambda)$	a Poisson distribution with parameter λ
$\Rightarrow$	weak convergence
N_t	number of particles alive in a process at time t
$A(t)$	the set of indices of alive particles at time t
$\preceq$	stochastically dominates

Abbreviations

BM	Brownian motion
BBM	branching Brownian motion

Acknowledgements

First, I thank my advisor, Jim Nolen, for all his work and support. In addition to suggesting these problems, he provided crucial guidance, ideas, and encouragement at all stages of this work. I am also grateful for the many useful and interesting discussions which helped me to develop as both a teacher and a mentor, and for the great example he was in both these areas. I am truly thankful for his mentorship.

I would also like to thank the other members of my committee: Jonathan Mattingly, Rick Durrett, and Mike Reed. I was incredibly lucky to have them as instructors and mentors in my research problems, general mathematics, and academic life.

I thank both the National Science Foundation and the Duke University Graduate School for financial support.

Finally, I would like to thank my family for their love and support. I especially thank my husband for the countless hours spent discussing my work and for his unwavering belief in me.

1

Introduction

An interacting particle system is a collection of particles that behave in a prescribed random way and interact with each other through a set of rules or through a common environment. Many of the mathematical questions which arise concerning interacting particle systems are related to how different microscopic rules of motion and interaction between particles influence the macroscopic system. This is a broad idea which can be captured in many different ways; relevant aspects of study include looking at what happens as the number of particles in the system is increased or analyzing the system behavior after a long period of time. In addition to the mathematical interest in these problems, interacting particle systems are appealing to a broader scientific audience as a way to model biological and physical observations. As such, the interplay between these subjects is large; biology and physics have provided substantial inspiration in the mathematical development of the field of interacting particle systems.

1.1 Evolutionary Models

Interacting particle systems have been used extensively to model evolutionary ideas of selection and competition. This is done primarily in one of two ways.

The first is through a selective force, which removes/kills particles from the system. One way this is invoked is to have a selection rule based on a particle's position in relation to other particles in the system. This selection dictates interaction between the particles and drives the evolution of the population.

Another method of evolutionary interaction focuses on a particle's reproduction rate, rather than removal of particles from the system. Such models assume that fitness of an individual can be represented by a change in the relative branch/birth rates of the particles; that is, the more fit individuals in a population are the ones that are more likely to reproduce. This interaction mechanism is common in multi-type branching models, especially those which model diseases with the ability to mutate, such as cancer. Formulating fitness in this way removes the need for a pre-defined "fitness landscape", which is a quantity that can be difficult to get a hold of in biological applications. As such, this formulation is often appealing to those who are looking to use experimental parameters in the construction of models.

1.2 Structure of Paper

In this dissertation, we will look at both of these types of competition through different interacting particle systems. The focus of the results is analysis of the long time limits and large particle limits of the processes.

In Chapter 2, we give necessary background to understand the work in later chapters, including definitions of the common objects and an introduction of theorems and properties that will be of use. In Chapter 3, we study the N-BBM branching-selection process with fitness function $x + \Psi(x)$, where $\Psi(x)$ is a periodic function. In Chapter

4, we give results for an N-BBM system where the fitness function is a symmetric, decreasing function with a single local maximum at the origin. Finally, in Chapter 5, we analyze a system of branching Brownian motions in which the particles interact through their branch rate. We study the case where the branch rate takes a specific form as a function of the empirical measure.

2

Background

2.1 Branching Brownian Motion

A fundamental component of each of the particle systems we study is an object called *branching Brownian motion (BBM)*. Informally, branching Brownian motion is a spatial stochastic process in which particles move like Brownian motions and at random, exponential times, split into a (possibly random) number of new, independent Brownian motions.

2.1.1 Brownian Motion

Branching Brownian motion is built out of a combination of individual Brownian motions. A one-dimensional *Brownian motion* is a stochastic process with the following properties

1. If $t_0 < t_1 < \cdots < t_n$, then $B(t_0), B(t_1) - B(t_0), B(t_2) - B(t_1), \ldots,$ $B(t_n) - B(t_{n-1})$ are independent.

2. If $s, t \geq 0$, then $B(t+s) - B(s) \overset{d}{=} N(0,t)$; that is,

$$\mathbb{P}(B(t+s) - B(s) \in A) = \int_A \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-x^2/2t}\, dx$$

3. $t \mapsto B(t)$ is continuous with probability 1

One can show that such a process exists (see for instance [12], among others). From the definition of one-dimensional Brownian motion, we can generalize to higher dimensions with multidimensional Brownian motion. A *d-dimensional Brownian motion* starting at $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ is a process $B(t) = (B^1(t), \ldots, B^d(t))$ where the $B^k(t)$'s are independent, one-dimensional Brownian motions with $B^k(0) = x_k$.

An *rate λ exponential random variable T* is a random variable with probability density function $f(x) = \lambda \mathrm{e}^{-\lambda x}$ for all $x \geq 0$ and $f(x) = 0$ for all $x < 0$. The associated CDF is $\mathbb{P}(T \leq x) = 1 - \mathrm{e}^{-\lambda x}$ for $x \geq 0$ and $\mathbb{E}[T] = 1/\lambda$. This random variable is important in the construction of branching Brownian motion because it is *memoryless*; that is, if T is a rate λ exponential random variable and $t, s \geq 0$, then

$$\mathbb{P}(T > t + s | T > s) = \mathrm{e}^{-\lambda t}$$

This means that exponential random variables are independent of their past behavior, making them very useful in the construction of Markov processes (for more details, see [12]).

Another property of exponential random variables which will be useful is their relationship to the Poisson random variable. Let $\{T_i\}_{i \in \mathbb{N}}$ be a collection of independent rate λ exponential random variables. Fix a $t > 0$ and let $N_t = \max\{k \mid \sum_{i=1}^k T_i < t\}$; then $N_t \overset{d}{=} \mathrm{Poi}(\lambda t)$.

Exponential random variables have the property that if $T_1, T_2, \cdots, T_N$ are independent rate λ exponential random variables, then $\min(T_1, T_2, \ldots, T_N)$ is equal in distribution to an exponential rate λN random variable.

2.1.2 Definition of Branching Brownian Motion

We will define a rate λ branching Brownian motion with offspring distribution ρ. Let $\{B_k(t)\}_{k\in\mathbb{N}}$ be a collection of independent Brownian motions in $\mathbb{R}^d$, starting at the origin, and $\{\tau_{jk}\}_{j,k\in\mathbb{N}}$ be a collection of independent exponential random variables with mean $\frac{1}{\lambda}$. Define ρ, a probability distribution on $\mathbb{N} = \{0, 1, 2, \cdots\}$ and let $\{A_k\}_{k\in\mathbb{N}}$ be a collection of independent random variables distributed according to the distribution ρ. For $t < \tau_{11}$, $X(t) = B_1(t)$. At $t = \tau_{11}$, $A_1 - 1$ new particles are added to the system; if $B_1(\tau_{11}) = b_1$, then we say that $X(\tau_{11}) = (b_1, b_1, \ldots, b_1) \in \mathbb{R}^{A_1}$. Each of the new particles will move like an independent Brownian motion. Let $m_2 = \min_{k=1,\ldots,A_1} \tau_{2k}$. For $\tau_{11} \leq t < m_2$, $X(t) = (B_1(t), b_1 + B_2(t - \tau_{11}), b_1 + B_3(t - \tau_{11}), \cdots, b_1 + B_{A_1}(t - \tau_{11}))$. At m_2, particle X_k branches and introduces $A_2 - 1$ new particles into the system, where $k = \arg\min_{1\leq j\leq A_1} \tau_{2j}$. Therefore, letting $X_k(m_2) = b_2$, we get $X(m_2) = (B_1(m_2), b_1 + B_2(m_2 - \tau_{11}), \cdots, b_1 + B_{A_1}(m_2 - \tau_{11}), b_2, b_2, \cdots, b_2) \in \mathbb{R}^{A_1+A_2-1}$. The pattern continues in this manner, with each particle that is introduced into the system being associated to a Brownian process to define the increments and a collection of exponential random variables to determine splitting times for that particle.

If $A_k = 2$ a.s. for all k, we call the BBM a *binary branching Brownian motion.*

We can also view BBM as a measure-valued process, with

$$\mu(t) = \sum_{k=1}^{N_t} \delta_{X_k(t)}$$

where N_t is the number of particles alive at time t (that is, number of particles whose birth time is before t).

2.1.3 Properties of Branching Brownian Motion

As a fundamental probabilistic object, the properties of branching Brownian motion are well studied. Some properties of this object are summarized below.

The first property gives us a distribution for the number of particles

Theorem 1. *Let N_t be number of particles in a binary BBM at time t. Then*

$$N_t \stackrel{d}{=} \mathrm{Geo}(\mathrm{e}^{-\lambda t})$$

That is, the distribution of N_t is geometric, with $\mathbb{P}(N_t = k) = (1 - \mathrm{e}^{-\lambda t})^{k-1}(\mathrm{e}^{-\lambda t})$.

The proof of this can be done by verifying that the probability generating function of the variable N_t is precisely that of a geometric random variable with the appropriate parameter. A proof in a restricted case can be found in Appendix A.

One of the tools most frequently used in the analysis of particle systems is the hydrodynamic limit. A *hydrodynamic limit* is a scaling limit of the microscopic system which reveals macroscopic properties, often as the solution to a partial differential equation (PDE). The name comes from hydrodynamics itself; particles move in a fluid randomly but display large-scale, deterministic behavior that can be quantified using PDEs. Finding a hydrodynamic limit can be thought of as trying to find a law of large numbers (LLN) for the empirical measure of the system. In the case of non-interacting particles, this is well-established; once the particles begin to interact and independence is lost, this LLN can be difficult to obtain.

The following theorem describes the hydrodynamic limit of a system of independent branching Brownian motions.

Theorem 2. *Let $X^N(t)$ be a particle system beginning with N binary, rate 1 branching Brownian motions in $\mathbb{R}$ where the initial positions of each particle are chosen independently and distributed according to the probability density $\rho(x)$. Let*

$\mu_t^N(x) = \frac{1}{N}\sum_{k=1}^{N_t} \delta_{X_k(t)}$, *where* N_t *is the number of particles alive at time* t.

Then $\mu_t^N(dx) \Rightarrow u(x,t)\,dx$ *weakly, where* $u(x,t)$ *is the solution to the pde*

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + u \quad x \in \mathbb{R}, t > 0 \\ u(x,0) &= \rho(x) \end{aligned} \tag{2.1}$$

The proof contains a classic argument in the study of hydrodynamic limits. In particular, we must argue two things. First, we show that the collection of solutions is tight and is therefore pre-compact. Second, we show that any limit of any subsequence is a weak solution to 2.1. The uniqueness of the solution to the PDE in turn guarantees uniqueness of the limit, giving the desired result. A full proof can be found in the appendix.

2.2 Interacting Particle Systems

As defined above, each particle in a branching Brownian motion is independent of the others. The process we now focus on includes particles which interact with each other. Adding dependence in the process makes the study of the system more complicated, and small variations to the system can make the process much less amenable to study.

2.2.1 Branching Brownian Motion with Selection

The interacting particle model that will be studied in Chapters 3 and 4 is from a class of models called *branching Brownian motion with selection.* These are branching processes in which interaction between the particles occurs through the removal of particles from the system, depending in some way on the other particles in the system. We will employ a selection mechanism which ensures that there are exactly N particles designated as alive at any time t; this process is often called an N-BBM, to indicate the fixed system size. One can interpret these processes as models of survival of the fittest, and this idea dictates much of the terminology surrounding the process.

2.2.2 Definition of an N-BBM

We define an *N-BBM with fitness function $V(x)$* as follows. First, we specify a multi-type branching process that will generate the N-BBM process. Let $\{Y_i\}_{i=1}^N$ be a collection of independent, binary, rate λ branching Brownian motions. At time 0, all particles are labeled as alive (A). Let τ_1 be the first branch time of the Y_i's. If $k = \arg\min_{1 \le i \le N} V(Y_i(\tau_1))$, where any ties are broken by uniformly choosing one particle index, then the label of Y_k changes to dead (D) at time τ_1. All particles which are born take the label of their parent, and all particles move like independent branching Brownian motions. Note that if Y_k is both the particle that branches and the particle that is removed, we choose to have the new particle born be alive. Therefore, the process at τ_1 contains N type A particles and 1 type D particle. The process is then defined in a similar manner for all t. At each time t, there will be N alive particles, and the number of dead particles will grow over time. For each time t, let $X_1(t), \cdots, X_N(t)$ enumerate the positions of the alive particles, with $X_1(t) \ge X_2(t) \ge \cdots \ge X_N(t)$. Then we define the N-BBM $X(t)$ as $(X_1(t), X_2(t), \ldots, X_N(t)) \in \mathbb{R}^d$.

Notice that the only way for a particle to be alive at time t is for the particle and all its ancestors to have been alive for all $s \le t$. That is, type D particles can never become type A particles.

Unless otherwise indicated, we will assume without loss of generality that the N-BBM is rate 1; that is, $\lambda = 1$.

2.2.3 Properties of an N-BBM

Notice that unlike BBM, there are always exactly N particles in N-BBM. Therefore, the waiting time between each birth event is an independent, rate $N\lambda$ exponential random variable (since it is the minimum of N independent rate λ exponential random variables). This leads us to a theorem about the behavior of N-BBM birth events.

Theorem 3. *Consider a time interval $[t, t+h]$ and let M_h be the number of birth events of a rate 1 N-BBM process in that interval. Then $M_h \stackrel{d}{=} \mathrm{Poi}(Nh)$ and therefore, $\mathbb{P}(M_h \geq 2) = O(h^2)$ as $h \to 0$. More precisely, this means that there exists a constant C such that*

$$\limsup_{h \to 0} \frac{\mathbb{P}(M_h \geq 2)}{h^2} \leq C$$

The first statement in the theorem follows from the fact that interarrival times in a Poisson point process on $\mathbb{R}^+$ are exponential, and the second part of the theorem follows from the first. Because $M_h \stackrel{d}{=} \mathrm{Poi}(Nh)$, we have that

$$\begin{aligned}
\mathbb{P}(M_h \geq 2) &= 1 - \mathrm{e}^{-Nh}(1 + Nh) \\
&= 1 - (1 + Nh) \sum_{k=0}^{\infty} \frac{(-Nh)^k}{k!} \\
&= (Nh)^2 - (1 + Nh) \sum_{k=2}^{\infty} \frac{(-Nh)^k}{k!} \\
&= \sum_{k=2}^{\infty} c_k h^k
\end{aligned}$$

for constants c_k. So this probability is $O(h^2)$ as $h \to 0$.

2.3 Additional Notation and Key Ideas

2.3.1 Ulam-Harris Notation for BBM

In addition to the notation for BBM used in 2.1.2, some proofs will also make use of the *Ulam-Harris labeling system*. This labeling system makes it easier to reference the underlying tree structure of BBM; it is defined as follows.

Let $Y(t)$ be a branching Brownian motion and $\mathcal{U}$ be defined as the set of all finite ordered tuples of the natural numbers:

$$\mathcal{U} = \bigcup_{i=1}^{\infty} \mathbb{N}^i$$

We associate to each particle in $Y(t)$ an element $u \in \mathcal{U}$; the index is assigned to indicate the lineage of the particle. That is, $Y_u(t)$ with $u = (1, 3, 2)$ is the location at time t of the second child of the third child of the first initial particle.

Those familiar with Ulam-Harris notation should note that this is slightly different than the standard definition. In particular, many uses of this labeling system include the label $\emptyset$ to indicate the initial particle at time 0 (corresponding to the root of the underlying Galton-Watson tree). However, because all the situations in which we will use this notation begin with not one but N branching Brownian motions, we omit the $\emptyset$ label and instead label the N initial particles with the tuples $\{(i)\}_{i=1}^{N}$.

Because the label of each particle relates directly to its lineage, the labels of particles give us more information about how particles relate to one another. For instance, Y_u is an ancestor of Y_v if there exists a $w \in \mathcal{U}$ such that $v = (u, w)$, where (u, w) is the concatenation of u and w. In this case, we write $u < v$. $|u|$, the number of coordinates in the tuple u, is the generation of Y_u.

When using this notation, we say that a parent particle dies and is replaced by their children at each birth event. If τ_v is the birth time of Y_v, then we need to define $Y_v(s)$ for $s < \tau_v$. We choose to use the convention that $Y_v(s) = Y_u(s)$, where $u < v$

and Y_u is alive at time s. This allows us to refer to the entire history of a particle through the positions of its ancestors.

When using Ulam-Harris notation, we will often write $A_t \subset \mathcal{U}$ to indicate the indices of the set of alive particles at time t. Therefore, $N_t = |A_t|$ would indicate the number of particles alive at time t. To avoid confusion, indices in Ulam-Harris notation will use the letters u, v, and w and indices in $\mathbb{N}$ as defined in Section 2.1.2 will use the letters i, j, and k.

2.3.2 Many-to-one Lemma

One of the basic facts which will be used extensively in this dissertation is a tool called the many-to-one lemma. We will state and prove the specific version used in the later proofs; however, much more general versions exist. See for instance, the statement and proof of a many-to-few lemma in [14].

Let $Y(t) = (Y_1(t), Y_2(t), \cdots, Y_{N_t}(t))$ be a rate λ branching Brownian motion with offspring distribution $\rho = \{\rho_m\}_{m\in\mathbb{N}}$. $N_t \in \mathbb{N}$ is the number of particles alive at time t and $Y(0) = (x)$.

Lemma 4 (Many-to-one Lemma)**.** *Let f be a measurable function on $\mathbb{R}$. Then*

$$\mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t))\right] = \mathbb{E}[N_t]\ \mathbb{E}_x[f(B(t))]$$

where $B(t)$ is an independent BM started at x.

Proof. We give a PDE proof of this fact. Define

$$u(x,t) = \mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t))\right]$$

and let τ be the first branch time of the BBM. Then we can say that

$$\begin{aligned} u(x,t) &= \mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau > t\right]\mathbb{P}(\tau > t) + \mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau \leq t\right]\mathbb{P}(\tau \leq t) \\ &= \mathbb{E}_x[f(B(t))](\mathrm{e}^{-\lambda t}) + \mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau \leq t\right]\mathbb{P}(\tau \leq t) \end{aligned}$$

Let $\eta(x,t) = \mathbb{E}_x[f(B(t)]$ and note that $\eta(x,t)$ solves

$$\eta_t = \frac{1}{2}\eta_{xx}$$

$$\eta(x,0) = f(x)$$

To simplify the second term, we condition on $\tau = s$:

$$\begin{aligned} \mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau \leq t\right]\mathbb{P}(\tau \leq t) &= \mathbb{P}(\tau \leq t)\int_0^t \mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau = s\right]\frac{\lambda\mathrm{e}^{-\lambda s}}{P(\tau \leq t)}\,ds \\ &= \int_0^t \lambda\mathrm{e}^{-\lambda s}\,\mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau = s\right] \end{aligned}$$

Define M to be the number of offspring at the branch event. Then

$$\mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau = s, M = m\right] = \mathbb{E}_x\left[\sum_{k=1}^{N_t^1} f(Y_k(t)) + \cdots + \sum_{k=1}^{N_t^m} f(Y_k(t)) \,\middle|\, \tau = s, M = m\right]$$

where N_t^j is the number of particles alive at time t whose ancestor was offspring j. Because each BBM is identical and independent, this gives the equivalence

$$\mathbb{E}_x\left[\sum_{k=1}^{N_t^1} f(Y_k(t)) + \cdots + \sum_{k=1}^{N_t^m} f(Y_k(t)) \,\middle|\, \tau = s, M = m\right] = m \cdot u(y, t-s)$$

Integrating over all possible branch locations $y \in \mathbb{R}$ and summing over the potential number of offspring, we get

$$\int_0^t \lambda \mathrm{e}^{-\lambda s} \mathbb{E}_x \left[\sum_{k=1}^{N_t} f(Y_k(t)) \,\middle|\, \tau = s \right] = \sum_m m\rho_m \int_0^t \int_{\mathbb{R}} \lambda \mathrm{e}^{-\lambda s} u(y, t-s)\Phi(x-y, s)\, dy\, ds$$
$$= \mathbb{E}[M] \int_0^t \int_{\mathbb{R}} \lambda \mathrm{e}^{-\lambda(t-s)} u(y, s)\Phi(x-y, t-s)\, dy\, ds \tag{2.2}$$

where the last equality comes from a change of variables, and Φ is defined as

$$\Phi(x,t) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-x^2/2t}$$

which is the fundamental solution to the PDE

$$\Phi_t = \frac{1}{2}\Phi_{xx}$$

Combining what we have, we see that

$$u(x,t) = \mathrm{e}^{-\lambda t}\eta(x,t) + \mathbb{E}[M] \int_0^t \int_{\mathbb{R}} \lambda \mathrm{e}^{-\lambda(t-s)} u(y,s)\Phi(x-y, t-s)\, dy\, ds$$
$$= \mathrm{e}^{-\lambda t} \left[\eta(x,t) + \mathbb{E}[M] \int_0^t \int_{\mathbb{R}} \lambda \mathrm{e}^{\lambda s} u(y,s)\Phi(x-y, t-s)\, dy\, ds \right]$$

Isolating the expression in the brackets, we see that

$$w(x,t) = \eta(x,t) + \mathbb{E}[M] \int_0^t \int_{\mathbb{R}} \lambda \mathrm{e}^{\lambda s} u(y,s)\Phi(x-y, t-s)\, dy\, ds$$

is the Duhamel formula for the solution of the PDE

$$w_t = \frac{1}{2} w_{xx} + \mathbb{E}[M]\lambda \mathrm{e}^{\lambda t} u(x,t)$$
$$w(x,0) = f(x)$$

Since $e^{\lambda t}u(x,t) = w(x,t)$, we see that

$$w_t = \lambda e^{\lambda t}u(x,t) + e^{\lambda t}u_t$$
$$w_{xx} = e^{\lambda t}u_{xx}$$

Plugging these in the PDE solved by w, we get

$$\lambda e^{\lambda t}u + e^{\lambda t}u_t = \frac{1}{2}e^{\lambda t}u_{xx} + \mathbb{E}[M]\lambda e^{\lambda t}u$$
$$e^{\lambda t}u_t = e^{\lambda t}\left[\frac{1}{2}u_{xx} + (\mathbb{E}[M]-1)\lambda u\right]$$
$$u_t = \frac{1}{2}u_{xx} + (\mathbb{E}[M]-1)\lambda u$$

Therefore, $u(x,t)$ solves the PDE

$$u_t = \frac{1}{2}u_{xx} + (\mathbb{E}[M]-1)\lambda\, u \tag{2.3}$$
$$u(x,0) = f(x) \tag{2.4}$$

If we actually solve this PDE, we see that $u(x,t) = e^{(\mathbb{E}[M]-1)\lambda t}\mathbb{E}_x[f(B(t))]$. Noting that $\mathbb{E}[N_t] = e^{(\mathbb{E}[M]-1)\lambda t}$, we have proved the many-to-one lemma:

$$\mathbb{E}_x\left[\sum_{k=1}^{N_t} f(Y_k(t))\right] = \mathbb{E}[N_t]\ \mathbb{E}_x[f(B(t))]$$

□

The statement of this lemma is similar to that of Wald's lemma, but it does not require independence of the random variables $Y_k(t)$. It is necessary to remove this condition from the statement, as the locations of particles in a branching Brownian motion are dependent through their common ancestors.

3

N-BBM with a Linear + Periodic Fitness Function

3.1 Introduction

Let $X(t)$ be an N-BBM with fitness function $V(x) = x + \Psi(x)$, with $\Psi(x)$ a 2π-periodic function (see 2.2.2 for a precise process definition). We define a *selection window*, L, for the function $V(x)$ as

$$L = \inf \left\{ d \,\middle|\, V(x) \leq V(y) \text{ for all } x, y \in \mathbb{R} \text{ such that } x - y > d \right\} \tag{3.1}$$

That is, the selection window L is the smallest distance which guarantees that if $X_k(t) - X_j(t) > L$, then $V(X_k(t)) - V(X_j(t)) > 0$. If $V(x)$ is monotonic, then $L = 0$; in general, however, we will be interested in choices of Ψ for which $L > 0$.

Remember that we choose to label the particles in order of decreasing position in $\mathbb{R}$ for all time:

$$X_1(t) \geq X_2(t) \geq \cdots \geq X_N(t)$$

3.1.1 Related Work

Early work on this system focused on discrete models; particles did not move between branch events and all the motion of the system came from the displacement between a particle and its parent. In 1997, Brunet and Derrida studied a stochastic selection system to investigate the effect of a cutoff on the velocity of a traveling wave [8]. In their model, N particles branched at discrete times, with offspring displaced from their parents and selected to favor overall motion to the right. They computationally studied the decrease in the velocity of the traveling wave caused by the finite system. They found the velocity to be slowed at a rate of $(\log N)^{-2}$. Bérard and Gouére [4] later verified this asymptotic rate for a similar class of particle systems.

Many variants of Brunet and Derrida's original system have been studied in recent years. Discrete models, without the added complication of movement between branch times, were the first tractable variations to be considered. One such variant was studied in 2011 by Durrett and Remenik [13]; they looked at a broad class of discrete-time models similar to those considered in [4], focusing on proving a hydrodynamic limit. They showed that the system has a positive limiting speed for large times and has a hydrodynamic limit; the empirical measure limits to an absolutely continuous measure solving a free boundary PDE as the number of particles goes to infinity.

N-BBM was introduced in 2014 by Berestycki and Zhao [6] as a continuous time variant of Brunet and Derrida's system. They studied the problem in $d \geq 1$ dimensions with fitness functions $V(x) = ||x||$ and $V(x) = \langle x, \nu \rangle$ for some vector ν. They were able to describe the long time speed and shape of the finite particle system. In 2017, De Masi, Ferrari, Presutti, and Soprano-Loto [11] proved a hydrodynamic limit of N-BBM for $d = 1$ and $V(x) = x$.

3.1.2 Contribution and Difficulties

In the work of Berestycki and Zhao [6] and of DeMasi et. al. [11], the arguments often rely on the monotonicity of the fitness function. That is, the ordering of particles by position and by fitness is the same. This is key in both the coupling arguments used and in meeting the conditions of the subadditive ergodic theorem (see Proposition

2 in [4] for an example of a classic proof technique). The speed of the system and the hydrodynamic limit have not been studied before for N-BBM in the absence of a monotonic fitness function; many of the standard proof techniques do not apply in this greater generality. The interplay of the non-monotonic fitness function, the continuous time motion of the particles, and the continuous time branching behavior distinguishes the problem studied here from previous work.

3.1.3 Motivation for a non-monotonic fitness function

For the remainder of this chapter, we consider an N-BBM system with a fitness function $V(x) = x + \Psi(x)$, for $\Psi(x)$ a 2π-periodic function. Depending on the choice of Ψ, $V(x)$ may not be monotonic. We are particularly interested in the idea of a fitness function with local fitness valleys: zones of lower fitness which the particles must cross to reach zones of higher fitness. Considering such a fitness function provides an intriguing mathematical setting and gives an abstract look at a situation of biological mutation which has been conjectured in the development of cancer [15]. Fitness landscapes with local maxima have been of theoretical and experimental interest to biologists since the 1930's when Sewall Wright studied the phenomenon and conjectured his Shifting Balance Theory as a mechanism for movement between local fitness maximums [18]. A possible factor behind the presences of fitness valleys in genetic landscapes is the idea of epistasis. *Epistasis* is the term given to the interactions between mutations. When mutations interact in a non-additive way, pathways of evolution between two high fitness states can be interspersed with low fitness states. In particular, biologists consider the case in which individuals have reached a local, but not global, fitness peak and ask whether populations can traverse the fitness valley to reach the global fitness peak (see [18] for a survey of the field from a biological perspective).

In this chapter, we study fitness functions with many local fitness peaks but no global maximum. The fitness function was not chosen to model a particular biological situation but rather to abstractly consider the idea of crossing fitness valleys in a mathematically interesting way. Because fitness functions are difficult to discover experimentally and likely are not as well-behaved as the functions chosen here, this work is not intended to be applied directly to a biological system. Rather, we study mathematically the question of how a population evolves in a landscape with regularly spaced local maxima, and observe through simulations the behavior where systems get stuck for a long time in a single fitness peak and the behavior where systems are able to travel quickly across multiple fitness peaks.

3.2 Main Results

First we show the existence of a stationary distribution of the system in a moving frame. From that result, it follows almost immediately that the system has a speed. To make these ideas precise, we define a shift random variable

$$k(t) = \arg\min_{k \in \mathbb{Z}} \{|X_N(t) - 2\pi k|\} \tag{3.2}$$

which tracks how many periods in front of (or behind) the origin the particle $X_N(t)$ is. In the case of multiple minimizing k values (which occurs when $X_N(t) = (2n+1)\pi$ for some integer n), we define $k(t)$ to be the smallest k value which minimizes that quantity. From that, we define the shifted process $Z(t)$

$$Z(t) = X(t) - 2\pi k(t) \tag{3.3}$$

Notice that $Z(t) \in [-\pi, \infty)^{N-1} \times [-\pi, \pi)$. Because the fitness function is unbounded, the moving frame is necessary to recenter the system over time. $Z(t)$ is a Markov process because determining the relative fitness values of the particles only depends on knowing their relative positions and their placement in the period of Ψ, and relative fitness is enough to decide which particle to remove.

Our first result says that $Z(t)$ is a positive recurrent Harris chain. Harris recurrence is the generalization of Markov chain recurrence to Markov processes with general state spaces; we defer the definitions to make this idea precise until Section 3.3.

Theorem 5. *$Z(t)$ is a positive recurrent Harris chain with a unique stationary distribution π.*

Once we have the existence of a stationary distribution of the shifted chain $Z(t)$, the speed of the process $X(t)$ follows from Birkhoff's ergodic theorem and bounds on the distance traveled by the maximum of N Brownian motions.

Theorem 6. *Let $X(0)$ be chosen according to π, the stationary distribution of $Z(t)$. Then the selection system, $X(t)$, has a speed. That is,*

$$\lim_{t\to\infty} \frac{X_N(t)}{t} = \gamma_N \qquad \textit{a.s and in } L^1 \tag{3.4}$$

for some constant γ_N.

Unfortunately, we do not know yet whether γ_N is strictly positive, though we conjecture that it is. See the further questions below and Remark 3.5 in Section 3.5 for more details on this conjecture.

The proof of Theorem 5 requires the following two technical lemmas. The first tells us that with high probability, the particles do not spread out too far. In particular, the statement concerns the distance between the first particle and the last particle. The proof relies on the fact that a single BBM cannot spread out too far; even though we have multiple branching Brownian motions and there is interaction, the spread of the particles can be related to the spread of a single BBM in a tangible way.

Lemma 7. *For all $\varepsilon > 0$ and $t > (1+\varepsilon)\ln N$,*

$$\mathbb{P}\left(|X_1(t) - X_N(t)| > 8\,(1+\varepsilon)\ln N + L\right) \le \frac{2}{N^{\varepsilon}} + \frac{C_\varepsilon}{N^{\varepsilon/2}} \tag{3.5}$$

In particular,

$$\lim_{N\to\infty} \mathbb{P}\left(|X_1(t) - X_N(t)| > 8\,(1+\varepsilon)\ln N + L\right) = 0 \tag{3.6}$$

Notice that the bound on the distance depends on N and ε but not on t. We have the explicit representation $C_\varepsilon = \frac{2+\varepsilon}{\varepsilon}$. Once we know that the particles are relatively close to each other with high probability, independent of t, we can choose an appropriate collection of sets to show that in the shifted system, the induced measures on $\mathbb{R}^N$ are tight.

Lemma 8. *$\{Z(t)\}_{t\in\mathbb{R}^+}$ is tight. That is, for every $\varepsilon > 0$, there exists a compact set K_ε such that $\mathbb{P}(Z(t) \in K_\varepsilon^c) < \varepsilon$ for all t.*

These lemmas are proven in Section 3.3. We end this introductory section with a few open questions.

Further Questions The results here imply only that the system has some speed, not that this speed is positive. We believe that this speed should be positive. In fact, we make the conjecture that γ_N is exactly the one-dimensional Brunet-Derrida particle speed, so mirroring a result from [6], we conjecture that as $N \to \infty$, $\gamma_N \to \sqrt{2}$. An intuitive argument for why this should be the limiting speed can be found in Remark 3.5 in Section 3.5.

Another open question is the convergence of the empirical distribution to a limiting, absolutely continuous measure whose density is a solution to a specific free boundary PDE.

Conjecture 9. *Let $\mu_t^N = \frac{1}{N}\sum_{k=1}^N \delta_{X_k(t)}$ be the empirical measure of the N-BBM process with fitness function $V(X) = x + \Psi(x)$ for Ψ a periodic function. Then $\mu_t^N(dx)$ converges weakly as $N \to \infty$ to a measure $u(x,t)\,dx$ in the space of measure-*

valued processes $D([0,T],\mathcal{M}_1)$ *and* $u(x,t)$ *is the solution to the free boundary PDE*

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + u \quad && x \in \mathbb{R}, t > 0 \\ \int_{\Omega_{\ell(t)}} u(x,t)\,dx &= 1 \quad && \textit{for all } t \\ u(x,t)|_{\partial\Omega_{\ell(t)}} &= 0 \quad && \textit{for all } t \end{aligned} \tag{3.7}$$

where such a solution exists.

The equation for u_t is the equation satisfied by a BBM with rate 1. The integral condition regulates the growth, which reflects the influence of the constant system size. However, because the particles can be moving, the domain of that integral can change, giving a free boundary condition. $V(x)$ is not explicitly apparent in the statement of this PDE, but $V(x)$ will influence the free boundary $\ell(t)$.

Computational simulations of the process support this conjecture, but we have been unable to prove it analytically. Computer-generated plots comparing the solution to the PDE and particle system simulations can be seen in Section 3.5.

3.3 Proof of Theorem 5

We are seeking to prove that the shifted chain $Z(t)$ is positive Harris recurrent. We begin with the proof of Lemma 7, which captures the idea that the alive particles do not spread out too far from one another. This is not immediately obvious. We know that the distance between the maximal and minimal particles in a standard branching Brownian motion at time t has order t. So without selection, the first and last particles drift farther apart as time goes on. Lemma 7 says that this is not true in this N-BBM system; there is probabilistic bound on the likelihood of the front and back particles being too far apart, where the spread is independent of t.

3.3.1 Spread of the Particles

Our first lemma concerns free binary BBM: branching Brownian motion with no selection. From the definition of N-BBM with selection given in 2.2.2, it is clear that for each N-BBM process, we have an associated free BBM process given by $Y(t) = (Y_1(t), \ldots, Y_N(t))$ where each Y_i is an independent binary, rate 1 BBM. This process simply ignores the type of each particle and only describes the underlying movement and branching structure. We prove a result about how long it takes a single binary BBM, $\hat{Y}(t)$, to grow to size M. We will use Ulam-Harris notation to refer to the particles in $\hat{Y}(t)$. Here, if τ_v is the birth time of particle $\hat{Y}_v$ and $u \leq v$ with $\tau_u < t$, then we define $\hat{Y}_v(t) = \hat{Y}_u(t)$ for $t < \tau_v$ (that is, when referring to a particle's position before its birth time, we are actually referring to the position of its ancestor alive at that time).

Lemma 10. *Let $\hat{Y}(t)$ be a free binary branching Brownian motion with branch rate λ. Let $\tau_k = \inf\{t \geq 0 \mid |\hat{Y}(t)| = k\}$ be the birth time of the kth particle. Then for any $M > 1$, $\varepsilon > 0$,*

$$\mathbb{P}\left(\tau_M > \frac{(1+\varepsilon)\ln M}{\lambda}\right) \leq \frac{C_\varepsilon}{M^{\varepsilon/2}}$$

where C_ε is a constant which depends only on ε.

Proof of Lemma 10. First, note that $\tau_{k+1} - \tau_k \stackrel{d}{=} \operatorname{Exp}(\lambda k)$ because there are k particles with independent rate λ branch clocks. With this in mind, we write

$$\tau_M = \sum_{k=1}^{M-1} \tau_{k+1} - \tau_k$$

where $\tau_1 = 0$. So τ_M is the sum of $M-1$ independent exponentials. This means that for $\theta < \lambda$, we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\theta\tau_M}\right] &= \prod_{k=1}^{M-1} \mathbb{E}\left[\mathrm{e}^{\theta(\tau_{k+1}-\tau_k)}\right] \\ &= \prod_{k=1}^{M-1} \frac{\lambda k}{\lambda k - \theta} \\ &\leq (M-1)\frac{\lambda}{\lambda - \theta} \\ &\leq \frac{M}{1-\theta/\lambda}\end{aligned}$$

where we have used the moment generating functions of the exponential random variables to get the second equality. Now we apply the exponential Chebyshev inequality:

$$\begin{aligned}\mathbb{P}\left(\tau_M > \frac{(1+\varepsilon)\ln M}{\lambda}\right) &\leq \mathrm{e}^{-\theta(1+\varepsilon)(\ln M)/\lambda}\mathbb{E}\left[\mathrm{e}^{\theta\tau_M}\right] \\ &\leq M^{-\theta(1+\varepsilon)/\lambda}\frac{M}{1-\theta/\lambda}\end{aligned}$$

Choosing θ such that $\theta/\lambda = \frac{2+\varepsilon}{2+2\varepsilon}$, we get that

$$\mathbb{P}\left(\tau_M > \frac{(1+\varepsilon)\ln M}{\lambda}\right) \leq C_\varepsilon M^{-\varepsilon/2}$$

where $C_\varepsilon = \frac{2+2\varepsilon}{\varepsilon}$. $\square$

This bound holds for all $\varepsilon > 0$ but is only useful for ε sufficiently large, in terms of M, because $C_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Now we are ready to prove that the particles do not spread out too much over time, i.e. no more than a constant which depends on N.

Proof of Lemma 7. We introduce the following quantities for ease of reference:

$$T = (1+\varepsilon)\ln N$$

$$R = 2T$$

Let $u_k(t)$ be the index of the kth rightmost type A particle at time t. Then for any $t \geq s$, we define

$$F_k^s(t) = \{u \in A_t \mid u_k(s) \leq v\}$$

The set $F_k^s(t)$ is the set of indices of the free particles at time t which are descendants of $Y_{u_k(s)}(s)$, the kth rightmost type A particle in $Y(s)$. This set allows us to keep track of both type A and D descendants of $X_k(s)$ at a later time t. Finally, define $\mathcal{N}^s(t)$ be the set of all indices of free particles at time t of type A or D which were offspring of particles alive at time s

$$\mathcal{N}^s(t) = \bigcup_{k=1}^{N} F_k^s(t)$$

Also of importance will be the events (recalling that L is the selection window for Φ defined in Section 3.1)

$$A_t = \{|X_1(t) - X_N(t)| > 4R + L\}$$

$$B_t = \left\{ \sup_{s\in[t-T,t]} |Y_u(s) - Y_u(t-T)| < R \text{ for all } u \in \mathcal{N}^{t-T}(t) \right\} \cap \left\{|F_1^{t-T}(t)| \geq N\right\}$$

which can be defined for all $t \geq T$. Event B_t is the event which bounds the movement of all particles at time t from its alive ancestor at time $t-T$ and specifies that the number of descendants of the leading type A particle at time $t-T$ has grown to size N by time t.

By the definition of R and A_t, it is clear that

$$\mathbb{P}(A_t) = \mathbb{P}\left(|X_1(t) - X_N(t)| \geq 8\,(1+\varepsilon)\ln N + L\right)$$

We will prove the lemma first by showing that $A_t \subseteq B_t^c$, and then giving a bound on $\mathbb{P}(B_t^c)$.

We will show that $\mathbb{P}(A_t \cap B_t) = 0$. When B_t occurs, each free particle satisfies $|Y_u(t) - Y_u(t-T)| \leq R$. Let $x_0 = X_1(t-T)$. When B_t occurs, each $Y_u(t)$ with $u \in \mathcal{N}^{t-T}(t)$ satisfies the relationship

$$\begin{aligned} Y_u(t) &\leq Y_u(t-T) + R \\ &\leq x_0 + R \end{aligned} \tag{3.8}$$

The second inequality follows from the definition of x_0, as it is the maximal type A particle at time $t - T$. Since $X_1(t)$ must be one of the particles $Y_u(t)$, Equation 3.8 also says

$$X_1(t) \leq x_0 + R \tag{3.9}$$

We will be done if we can bound $X_N(t)$ from below. To do this, we break the possible process behaviors into two cases. Let $t^* = \min\{s > t - T \mid |F_1^{t-T}(s)| \geq N\}$. Notice that when B_t occurs, $t^* \leq t$.

Case 1: If $X_N(s) < \min_{u \in F_1^{t-T}(s)} Y_u(s) - L$ for all $s \in [t-T, t^*)$, then no offspring of $Y_{u_1(t-T)}(t-T)$ is removed by selection before time t^*. Since $|F_1^{t-T}(t^*)| = N$ and none of these particles were selected before time t^*, we know that all type A particles at time t^* must be descendants of $X_1(t-T)$. Because type A particles can only come from a type A ancestor, this means that all the type A particles at time t are also descendants of $X_1(t-T)$. By the definition of B_t, all the offspring of $Y_{u_1(t-T)}(t-T)$ must remain within R of $Y_{u_1(t-T)}(t-T)$. Therefore, $X_N(t)$, a type A particle at time t, must be bounded below by

$$\begin{aligned} X_N(t) &\geq Y_{u_1(t-T)}(t-T) - R \\ &= x_0 - R \end{aligned} \tag{3.10}$$

Combining 3.10 with 3.9, we get the bound

$$|X_1(t) - X_N(t)| \leq 2R \tag{3.11}$$

Therefore, in this case, $|X_1 - X_N| < 4R + L$, so A_t and B_t cannot occur at the same time in this case.

Case 2: Suppose there exists an $s \in [t-T, t^*)$ such that $X_N(s) \geq \min_{u \in F_1^{t-T}(s)} Y_u(s) - L$; then selection from offspring of $X_1(t-T)$ can occur before t^*. Define

$$s^* = \min\{s > t - T \mid X_N(s) \geq \min_{u \in F_1^{t-T}(s)} Y_u(s) - L\}$$

Then,

$$\begin{aligned} X_N(s^*) &\geq \min_{u \in F_1^{t-T}(s)} Y_u(s) - L \\ &\geq x_0 - R - L \end{aligned} \tag{3.12}$$

We obtain the second inequality above by noting that every particle with an index in F_1^{t-T} is a descendant of $X_1(t-T)$, so given that B_t occurs, it can be no more than R away from x_0 at time s^*.

To extend this bound to time t, notice that any type A particle at time t is a descendant of a type A particle at time s^*. The leftmost type A particle at time s^* is $X_N(s^*)$ and satisfies the inequality $X_N(s^*) \geq x_0 - R - L$. Due to the restriction that children travel no more than R from their $t-T$ ancestors, $Y_{u_N(s^*)}(t-T) \geq x_0 - 2R - L$. Therefore all descendants of alive particles at time s^* had ancestors at time $t - T$ with positions to the right of $x_0 - 2R - L$. So because $X_N(t)$ must be a descendant of one of these alive particles at time s^*,

$$X_N(t) \geq x_0 - 3R - L \tag{3.13}$$

We combine this bound with 3.9 to get

$$|X_1(t) - X_N(t)| \leq 4R + L \tag{3.14}$$

Again, if $|X_1(t) - X_N(t)| \leq 4R + L$, A_t cannot have occurred, so $A_t \cap B_t$ is the empty set in this case as well. Therefore, $A_t \cap B_t = \emptyset$, so $\mathbb{P}(A_t) \leq \mathbb{P}(B_t^c)$.

We now find a bound on $P(B_t^c)$.

$$\begin{aligned} \mathbb{P}(B_t^c) &\leq \mathbb{P}(|F_1^{t-T}(t)| < N) \\ &\quad + \mathbb{P}\left(\sup_{s \in [t-T,t]} |Y_u(s) - Y_u(t-T)| \geq R \text{ for some } u \in \mathcal{N}^{t-T}(t) \right) \end{aligned} \tag{3.15}$$

Noticing that $\mathbb{P}(|F_1^{t-T}(t)| < N) = \mathbb{P}(t^* > T)$, we can use Lemma 10 to see that

$$\mathbb{P}(|F_1^{t-T}(t)| < N) \leq \frac{C_\varepsilon}{N^{\varepsilon/2}} \tag{3.16}$$

To bound the final term in 3.15, let $B(t)$ be a BM started at 0 and write

$$\begin{aligned} &\mathbb{P}\left(\sup_{s \in [t-T,t]} |Y_u(s) - Y_u(t-T)| \geq R \text{ for some } u \in \mathcal{N}^{t-T}(t) \right) \\ &\leq 2\mathbb{E}\left[\sum_{u \in \mathcal{N}^{t-T}(t)} \mathbb{1}\left\{ \sup_{s \leq T} |Y_u(s) - Y_u(t-T)| > R \right\} \right] \\ &= 2\mathbb{E}[\mathcal{N}^{t-T}(t)]\ \mathbb{P}\left(\sup_{s \leq T} B(t) > R \right) \\ &= 4\mathbb{E}[\mathcal{N}^{t-T}(t)]\ \mathbb{P}(B(T) > R) \\ &= 4N\mathrm{e}^T \mathbb{P}(B(T) > R) \\ &\leq 2N^{2+\varepsilon} \cdot N^{-2(1+\varepsilon)} \\ &= 2N^{-\varepsilon} \end{aligned}$$

where we have use the reflection principle and the last inequality is obtained from a common Gaussian tail bound (see Appendix A, Theorem 38). So

$$\mathbb{P}\left(\sup_{s\in[t-T,t]} |Y_u(s) - Y_u(t-T)| \geq R \text{ for some } u \in \mathcal{N}^{t-T}(t)\right) \leq 2N^{-\varepsilon} \tag{3.17}$$

Combining the fact that $\mathbb{P}(A_t) \leq \mathbb{P}(B_t^c)$ and Inequalities 3.15, 3.16, and 3.17, we see that

$$\mathbb{P}(A_t) \leq \frac{2}{N^\varepsilon} + \frac{C_\varepsilon}{N^{\varepsilon/2}} \tag{3.18}$$

Notice that $\lim_{N\to\infty} \mathbb{P}(A_t) = 0$ as desired. This concludes the proof of Lemma 7. □

3.3.2 Tightness of Shifted Process

Next, we want to show tightness of the measures induced by the shifted system $Z(t)$; that is, we want to find a compact set K_ξ such that for each t, $\mathbb{P}(Z(t) \in K_\xi) \geq 1 - \xi$ for any $\xi > 0$.

Proof of Lemma 8. Fix $\xi > 0$ and $N \geq 2$. Choose η to be sufficiently large such that

$$\frac{2}{N^\varepsilon} + \frac{C_\eta}{N^{\eta/2}} < \frac{\xi}{2} \tag{3.19}$$

We know that this can be done because the constant C_η gets smaller as η gets larger. We define

$$M = 8\,(1+\eta)\ln N + L \tag{3.20}$$

and $K_1 = [-\pi, M+\pi]^N$. Because $Z(t)$ preserves the order of the particles in $X(t)$,

$$\mathbb{P}(Z(t) \in K_1) = 1 - \mathbb{P}(|X_1(t) - X_N(t)| > M)$$

Applying the previous theorem, we can see that for all $t > (1+\eta)\ln N$,

$$\mathbb{P}(|X_1(t) - X_N(t)| > M) \leq \frac{\xi}{2} < \xi$$

so $\mathbb{P}(Z(t) \in K_1) \geq 1 - \xi$ for all $t > (1+\eta)\ln(N)$. To consider $t \leq (1+\eta)\ln N$, we define a new compact set $K_2 = [-\pi, M + 2I_\xi + \pi]^N$ where I_ξ is chosen so that

$$\mathbb{P}(X_1(0) > I_\xi) + \mathbb{P}(X_N(0) < -I_\xi) \leq \frac{\xi}{2} \tag{3.21}$$

Then we have that

$$\begin{aligned}\mathbb{P}(|X_1(t) - X_N(t)| >& M + 2I_\xi) \leq \mathbb{P}(|X_1(t) - X_N(t)| > \frac{M-L}{2} + 2I_\xi) \\ &\leq \mathbb{P}(|X_1(0) - X_N(0)| > 2I_\xi) \\ &+ \mathbb{P}\left(|X_1(t) - X_N(t)| > \frac{M-L}{2} + 2I_\xi \middle| |X_1(0) - X_N(0)| \leq 2I_\xi\right)\end{aligned}$$

If all particles stay within $2(1+\eta)\ln(N)$ of their original positions up to time $(1+\eta)\ln(N)$ and all particles started within $2I_\xi$ of each other, then $|X_1(t) - X_N(t)| \leq 4(1+\eta)\ln(N) + 2I_\xi = \frac{M-L}{2} + 2I_\xi$. Therefore, we can bound the second probability from above by the probability that at least one of the N BBMs has a particle leave the tube of radius $2(1+\eta)\ln(N)$. Using an identical calculation to the one used to get Equation 3.17, we can see that this probability is therefore bounded by $2N^{-\eta}$ as before.

$$\begin{aligned}\mathbb{P}(|X_1(t) - X_N(t)| > M + 2I_\xi) &\leq \mathbb{P}(|X_1(0) - X_N(0)| > 2I_\xi) \\ &+ \mathbb{P}\left(|X_1(t) - X_N(t)| > \frac{M-L}{2} + 2I_\xi \middle| |X_1(0) - X_N(0)| \leq 2I_\xi\right) \\ &\leq \frac{\xi}{2} + 2N^{-\eta} \\ &\leq \frac{\xi}{2} + \frac{\xi}{2} = \xi\end{aligned}$$

where the last inequality comes from our initial choice of η. Therefore,

$$\mathbb{P}(Z(t) \in K_2) > 1 - \xi$$

for $t \leq (1+\eta)\ln(N)$.

Because the union of compact sets is compact, we let $K_\xi = K_1 \cup K_2 = [-\pi, M + 2I_\xi + \pi]$, and we get that for all t,

$$\mathbb{P}(Z(t) \in K_\xi) \geq 1 - \xi$$

Therefore, the collection of random variables $\{Z(t)\}_t$ is tight. $\square$

3.3.3 Harris Chain Definitions and Proof

We continue towards the proof $Z(t)$ is a positive recurrent Harris chain. First, we give the necessary definitions, including defining precisely Harris recurrence, positive Harris recurrence, and petite sets - special sets which are representative enough that determining Harris recurrence on that set gives Harris recurrence of the process. We use the notation of [17].

Let $S(t)$ be a time homogeneous Markov process with state space $(X, \mathcal{B})$ with transition semigroup P^t. Suppose the process evolves on the probability space $(\Omega, \mathcal{F}, P_x)$, where $S(0) = x \in X$. For each measurable set A, we define

$$\eta_A = \int_0^\infty \mathbb{1}_{S(t)\in A}\, dt \tag{3.22}$$

If there exists a finite measure φ such that the event $\{\eta_A = \infty\}$ holds a.s. for all A with $\varphi(A) > 0$, then Z is called *Harris recurrent*. We use the idea of Harris recurrence as a way of making precise the idea of recurrence for a Markov chain in an uncountably infinite state space, like $\mathbb{R}^N$. η_A is a random variable defined as the amount of time $S(t)$ spends in the set A.

It is known that a Harris recurrent right process has an essentially unique invariant measure. If this invariant measure π is a finite measure, then we call S *positive Harris recurrent*.

The work of Meyn and Tweedie [17] gives two useful characterizations of Harris recurrence, which we will use when we get to the proof of Theorem 5 below. These characterizations require the idea of a sampled chain and the associated petite sets. We define only a very special case of the sampled chains discussed in [17]. A *T-skeleton chain* of $S(t)$ is the chain $S_k = S(kT)$ for some fixed time T. A *φ-petite set* for a T-skeleton chain is a set $A \in \mathcal{B}$ such that there exists a non-trivial measure φ with $P_T(x, \cdot) \geq \varphi(\cdot)$ for all $x \in A$.

We now put these definitions in context for the shifted process $Z(t)$. We will reference several theorems from Meyn and Tweedie [17], which are also stated in Appendix A for ease of reference. The state space of $Z(t)$ is $[-\pi, \infty)^{N-1} \times [-\pi, \pi)$. Fix an $\xi \in (0, 1)$ and define η as in the proof of Lemma 8. Define Z_k to be the discrete time T-skeleton chain of $Z(t)$ with $T = (1 + \eta) \ln N$. That is, $Z_k = Z(kT)$ for all $k \in \mathbb{Z}^+$. We show a petite set for this sampled chain.

Lemma 11. *Let $C_\xi = M + 2I_\xi$ as defined above in 3.19, 3.20, 3.21. Then $\hat{K}_\xi = [-\pi, C_\xi + \pi]^{N-1} \times [-\pi, \pi)$ is φ-petite with $\varphi(dx) = p\, dx$ on C and 0 on $\hat{K}_\xi^c$, where*

$$p = \mathrm{e}^{-TN} \left[\inf_{(x,y) \in [-\pi, C_\xi + \pi]^2} \Phi(x - y, T) \right]^N > 0$$

and $\Phi(x - y, T) = \frac{1}{\sqrt{2\pi T}} \mathrm{e}^{-(x-y)^2/2T}$ is the transition density for a single one-dimensional BM run for time T, starting from x.

Proof. We need to show that $P_t(\mathbf{x}, A) \geq \varphi(A)$ for all $\mathbf{x} \in \hat{K}_\xi, A \in \mathcal{B}$. Certainly for $A \cap \hat{K}_\xi^c$, this is true, as the measure φ is 0. Therefore, it suffices to show this for a measurable set $A \subset \hat{K}_\xi$. Let $\mathbf{x} \in \hat{K}_\xi$ be our starting position with $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ and let $\mu(A)$ be the Lebesgue measure of A. Then we know that

$$
\begin{aligned}
P_T(\mathbf{x},A) &\geq \mathbb{P}(\text{no branches before time } T \text{ and } X(t) \text{ moves from } \mathbf{x}\to A)\\
&= \mathrm{e}^{-NT}\cdot \mathbb{P}(\text{an } N\text{-dim. BM moves from } \mathbf{x}\to A)\\
&= \mathrm{e}^{-NT}\cdot \int_A \Pi_{i=1}^N \Phi(x_i-y_i,T)\,dy\\
&\geq \mathrm{e}^{-NT}\cdot \left(\inf_{(x,y)\in[-\pi,C_\xi+\pi]^2}\Phi(x-y,T)\right)^N \int_A dy\\
&\geq \mathrm{e}^{-NT}\left(\inf_{(x,y)\in[-\pi,C_\xi+\pi]^2}\Phi(x-y,T)\right)^N \mu(A)\\
&= p\mu(A)\\
&= \varphi(A)
\end{aligned}
$$

This shows that $\hat{K}_\xi$ satisfies the definition of a φ-petite set. □

Using this petite set, we can show that $Z(t)$ is positive Harris recurrent.

Proof of Theorem 5. Theorem 3.3 of [17] states that if $\hat{K}_\xi$ is a petite set and $P_x(\tau_{\hat{K}_\xi} < \infty) = 1$ for all $x \in [-\pi,\infty)^{N-1}\times[-\pi,\pi)$, where $\tau_{\hat{K}\xi} = \inf\{k \mid Z_k \in \hat{K}_\xi\}$, then $Z(t)$ is Harris recurrent. Here we are using the fact that the first hitting time of $\hat{K}_\xi$ by $Z(t)$ is bounded above by $T\tau_{\hat{K}_\xi}$, so it is enough to bound the expectation and probability of the hitting time by Z_k. By Lemma 7, we can see that the probability of being outside that $\hat{K}_\xi$ at time kT is bounded from above by ξ. So we can say that $\tau_{\hat{K}_\xi}$ is stochastically dominated by a geometric random variable: $\tau_{\hat{K}_\xi} \preceq G$ where $G \overset{d}{=} \mathrm{Geo}(1-\xi)$ because the chance the process is outside K_ξ at any time t is bounded above by ξ. Therefore, for all x, $\mathbb{P}_x(\tau_{K_\xi} < \infty) = 1$. Therefore, $Z(t)$ is Harris recurrent by Theorem 3.3 of [17].

To say that this chain is positive Harris recurrent, we use Theorem 1.2(a) of [17], which says that $Z(t)$ is positive Harris recurrent if and only if there exists a closed petite set C such that for some $\delta > 0$,

$$\sup_{x \in C} \mathbb{E}_x[\tau_C(\delta)] < \infty \tag{3.23}$$

where $\tau_C(\delta)$ is the first return time to C after time δ.

Notice that $\hat{K}_\xi$ is closed in the state space of $Z(t)$. Because our bound on the probability that the process is outside of $\hat{K}_\xi$ at time t does not depend on the starting configuration, we can use the same stochastic dominance in this case as above to say that

$$\sup_{x} \mathbb{E}_x[\tau_{\hat{K}_\xi}] < \infty \tag{3.24}$$

for all $x \in [-\pi, \infty)^{N-1} \times [-\pi, \pi)$. This means that

$$\sup_{x \in \hat{K}_\xi} \mathbb{E}_x[\tau_{\hat{K}_\xi}(1)] < \infty \tag{3.25}$$

Therefore, $Z(t)$ is positive Harris recurrent, which implies that $Z(t)$ has a unique invariant distribution. □

3.4 Proof of Theorem 6

Now we want to show that the existence of a stationary distribution for $Z(t)$ implies there is speed for the system. We will use Birkhoff's ergodic theorem to make a statement about the speed of the system.

Proof. Recall that we have defined $Z_N(t)$ as the last component of the process $Z(t)$ and $k(t)$ as the process which keeps track of the true spatial location of the process at time t. For any t, we can write

$$\begin{aligned} \frac{X_N(t)}{t} &= \frac{Z_N(t) + 2\pi k(t)}{t} \\ &= \frac{Z_N(t)}{t} + 2\pi \frac{k(t) - k(\tau_{m(t)})}{t} + 2\pi \frac{\sum_{j=1}^{m(t)} k(\tau_j) - k(\tau_{j-1})}{t} \end{aligned} \tag{3.26}$$

where the τ_j's are the branch times of $X(t)$ and $m(t)$ is the total number of branch events up to time t. So $\tau_j - \tau_{j-1} \stackrel{d}{=} Exp(N)$ and $m(t) \stackrel{d}{=} Poi(Nt)$. We let $\tau_{m(t)}$ be the most recent branching time before time t. Notice that because the distribution of $k(\tau_j) - k(\tau_{j-1})$ depends only on $Z(\tau_{j-1})$ and we know that the distribution of $Z(t)$ the stationary distribution for all times, the distribution of $k(\tau_j) - k(\tau_{j-1})$ is the same for all j. This means that each term in the sum is identically distributed. So we can apply Birkhoff's ergodic theorem to say that

$$\begin{aligned}
\lim_{t\to\infty} \frac{X_N(t)}{t} &= \lim_{t\to\infty} \frac{Z_N(t)}{t} + 2\pi \frac{k(t) - k(\tau_{m(t)})}{t} + 2\pi \frac{\sum_{j=1}^{m(t)} k(\tau_j) - k(\tau_{j-1})}{t} \\
&= 0 + \lim_{t\to\infty} 2\pi \frac{k(t) - k(\tau_{m(t)})}{t} + 2\pi \mathbb{E}_\pi[k(\tau_1) - k(0)] \lim_{t\to\infty} \frac{m(t)}{t} \\
&= 0 + 2\pi N \mathbb{E}_\pi[k(\tau_1)] \quad \text{a.s and in L}^1
\end{aligned} \tag{3.27}$$

The limit of the first term is 0 because $Z_N(t) \in [-\pi, \pi]$. Because $k(\tau_1) - k(0)$ can be bounded by the number of 2π-increments traveled by the maximum of N BMs in an exponential amount of time, we can use the fact that the maximum is exponentially unlikely to travel more than $\sqrt{2}\tau_1 + a$ where $\tau_1 \stackrel{d}{=} \text{Exp}(N)$ to say that $\mathbb{E}_\pi[k(\tau_1)] < \infty$. That fact also allows us to say that because $k(t) - k(\tau_{m(t)}) \leq \sup_{\tau_{m(t)} \leq s \leq \tau_{m(t)+1}} k(s) - k(\tau_{m(t)})$, and this supremum is summable by the same argument, the limit of the second term also goes to 0. The last equality also relies on the fact that $m(t)$ is Poisson with mean Nt, so $m(t)/t$ converges a.s. and in L^1 to N (see Appendix A). Therefore, we have the following limit

$$\begin{aligned}
\lim_{t\to\infty} \frac{X_N(t)}{t} &= 2\pi N \mathbb{E}_\pi[k(\tau_1)] \\
&= \gamma_N \quad \text{a.s and in } L^1
\end{aligned} \tag{3.28}$$

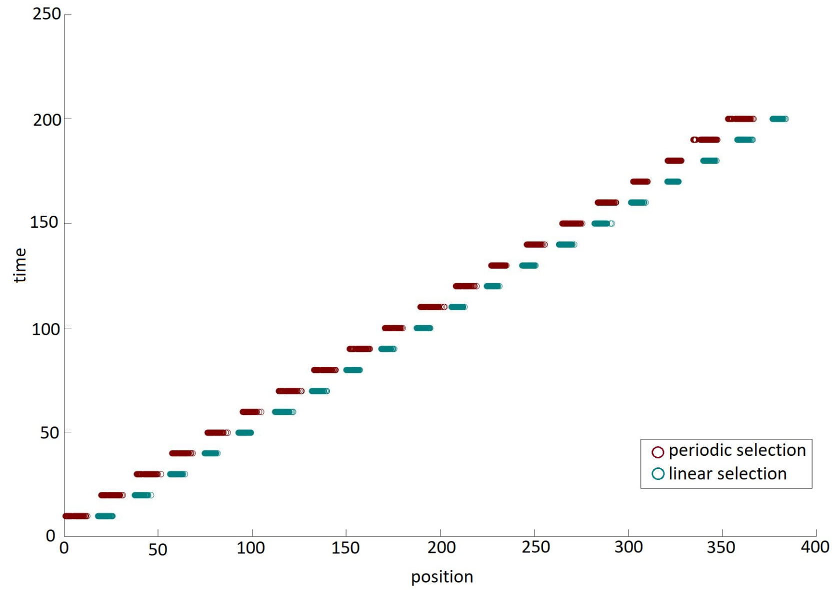
for some $\gamma_N < \infty$. Therefore, we have shown that the system has a speed. □

3.5 Computational Results

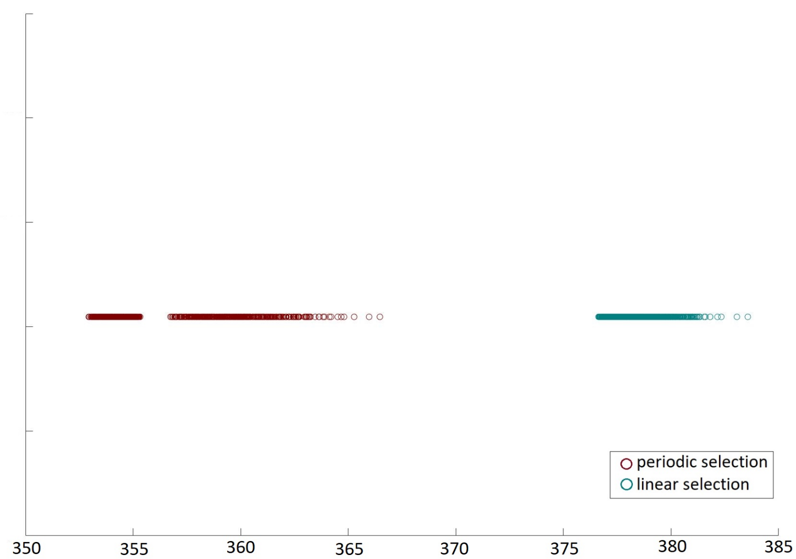
We will use the specific case of $\Psi(x) = 2\sin(x)$ to demonstrate several aspects of the system, with simulations done in MATLAB R2016a. In this case, $V(x)$ is not monotonic, and for this choice of $V(x)$, the finite particle system displays two distinct types of behavior: either the system moves through space, or the system becomes trapped in the peak of a local fitness maximum. We will look at each of these types of behavior separately. Note that the simulations were done using rate $\sqrt{2}$ BMs.

Moving Behavior In some simulations, the particles move to the right at a speed comparable to the speed of a linearly selected system. Figure 3.1a shows a comparison between particles selected according to $V(x) = x + 2\sin(x)$ (periodic selection), and particles selected according to $V(x) = x$ (linear selection). In this simulation, the birth times of the particles are coupled, which makes the comparison easier to visualize. Particles are all started from $x = 2$. Figure 3.1b shows the position of the particles when the simulation ended. In this figure, a gap occurs in the positions of particles subject to periodic selection. This split is positioned around a local fitness minimum. For a closer view of this phenomenon, see Figure 3.2. Both selection types have a similar decrease in the density of particles at the front of the system.

If we consider a 2-dimensional version, with the radially-symmetric fitness function $V_2(x) = ||x|| + 2\sin(||x||)$, we see similar results. We start all particles at (0,0) and set the branch rate $\lambda = .65$. If you view the particles at fixed time intervals, you can see that there are two groups of particles, with an unoccupied region in between (see Figure 3.3). This unoccupied region runs along an arc, because the level sets of $V_2(x)$ are circles. The particles themselves also appear to spread out along level sets, which matches the behavior of particles subject to the the fitness function $||x||$, as described by Berestycki and Zhao in [6]. Other similarities to the 2-dimensional



(a) A comparison of periodic selection and linear selection.



(b) The final positions of the particles, after 200,000 birth events.

FIGURE 3.1: In this simulation, $N = 1000, \lambda = 1$, and the positions of the particles are plotted every $10N$ birth events, up to 200,000 births. The processes are coupled through the birth times.

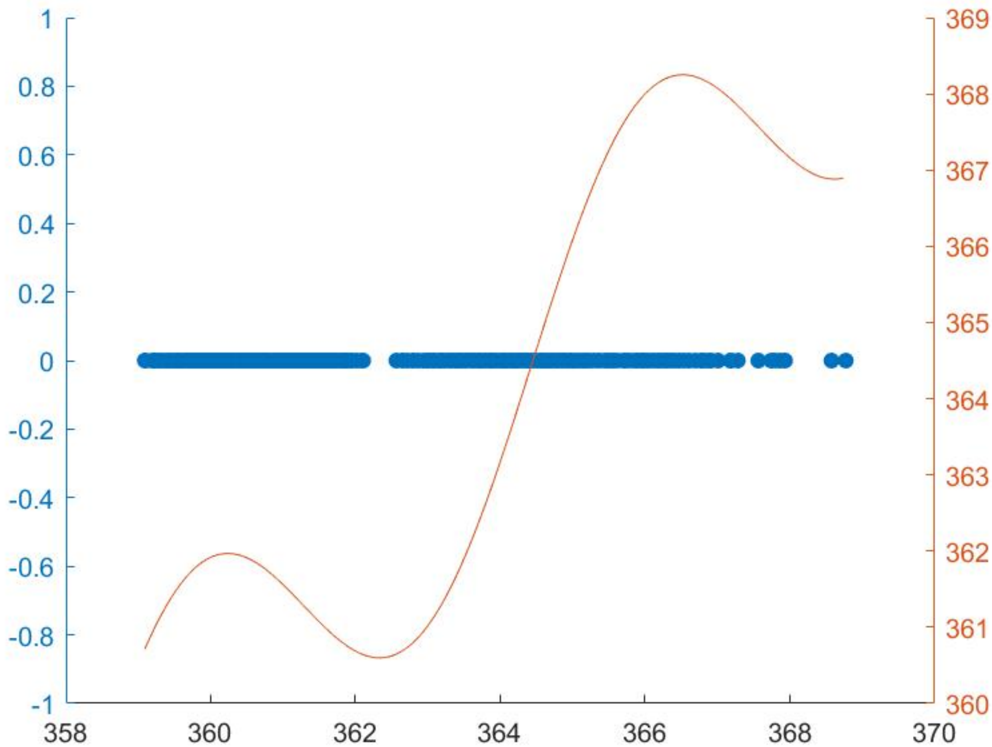


FIGURE 3.2: When $N = 1000, \lambda = 1$, after 200,000 birth events, the particles have a gap at the position of the corresponding fitness minimum. The fitness function (plotted) is $x + 2\sin(x)$.

results of Berestycki and Zhao include the eventual propagation of the particles in a fixed angular direction, which from simulation seems to be chosen uniformly at random from $[0, 2\pi)$.

Remark 3.5 The moving behavior of the periodic selection system appears to be very close to the speed of the linear selection system in many simulations. To give intuition behind this observation, consider the following heuristic argument. BBM with N particles has a spread on the order of $\ln(N)$, and since the selection window L is constant in N, once the particles have spread out, the particles outside the selection window feel essentially monotonic selection. That is, they only feel selection pressure

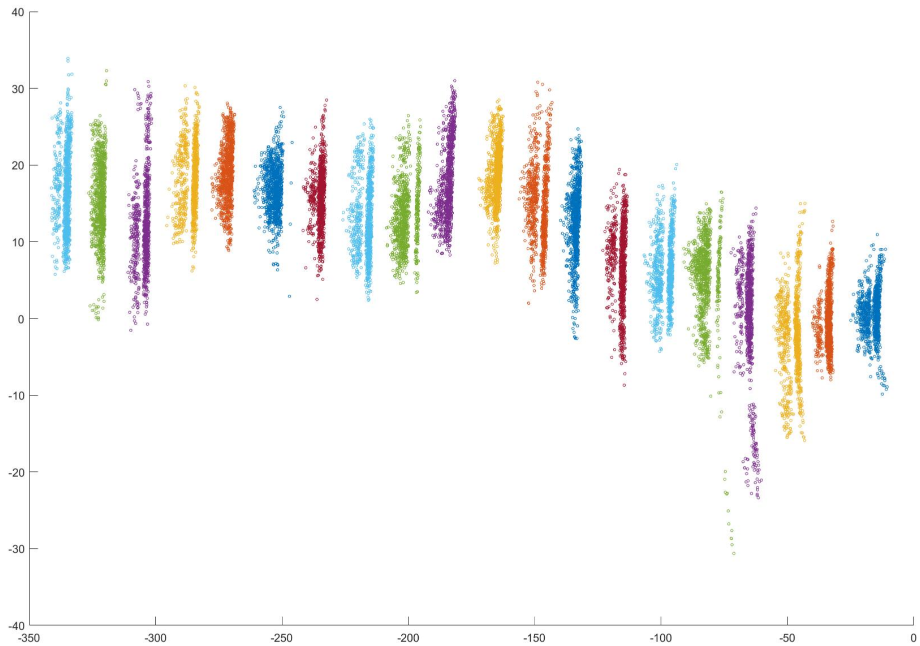


FIGURE 3.3: A simulation in two dimensions with $N = 1000$, $\lambda = .65$. The colors represent different stopping times in the simulation, with positions plotted every $10N$ birth events. The fitness function used here is $V(x) = ||x|| + 2\sin(||x||)$.

from behind them, and they move at a speed similar to the one-dimensional Brunet-Derrida particle speed, with a slightly different finite N correction. If $o(N)$ particles are found in the selection window as $N \to \infty$, then the finite N correction should be the same up to $o((\log N)^{-2})$. While a direct coupling cannot be established to prove this because of the nonzero probability that all particles are within the selection window, this intuition leads us to believe that as $N \to \infty$, $\gamma_N \to \sqrt{2}$ as is found in systems subject to monotonic selection.

Trapped Behavior For some simulations, instead of moving in any direction, the particles get stuck at the first local fitness maximum they encounter. In one dimension, the particles remain close to zero, and the histogram density plot in Figure 3.4 gives

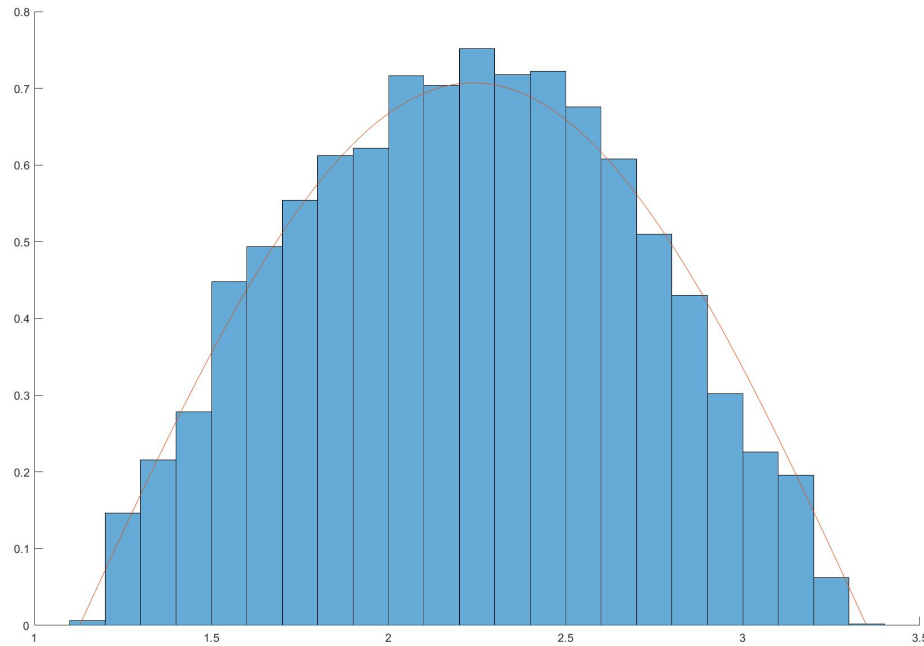


FIGURE 3.4: In this simulation, $N = 5000$ and $\lambda = 2$. The particles have remained trapped near the origin for 200,000 birth events. The histogram shows the density of the final particle positions. The line is the conjectured stationary distribution.

an example of the distribution of the particles in this case.

In two dimensions, the particles form an annulus around the origin, as seen in Figure 3.5. The colors represent different times at which the positions were plotted.

Conjecture After observing the behavior of the particles in many simulations, we made the conjecture that in one dimension, $\mu_t^N(dx)$, the empirical measure of the system, is a finite particle approximation of the measure $u(x,t)\,dx$ where $u(x,t)$ satisfies the PDE

$$\begin{aligned} u_t &= \Delta u + \lambda u \\ \int_{\Omega_{\ell(t)}} u(x,t)\,dx &= 1 \quad \text{for all } t \\ u(x,t)|_{\partial\Omega_{\ell(t)}} &= 0 \quad \text{for all } t \end{aligned} \tag{3.29}$$

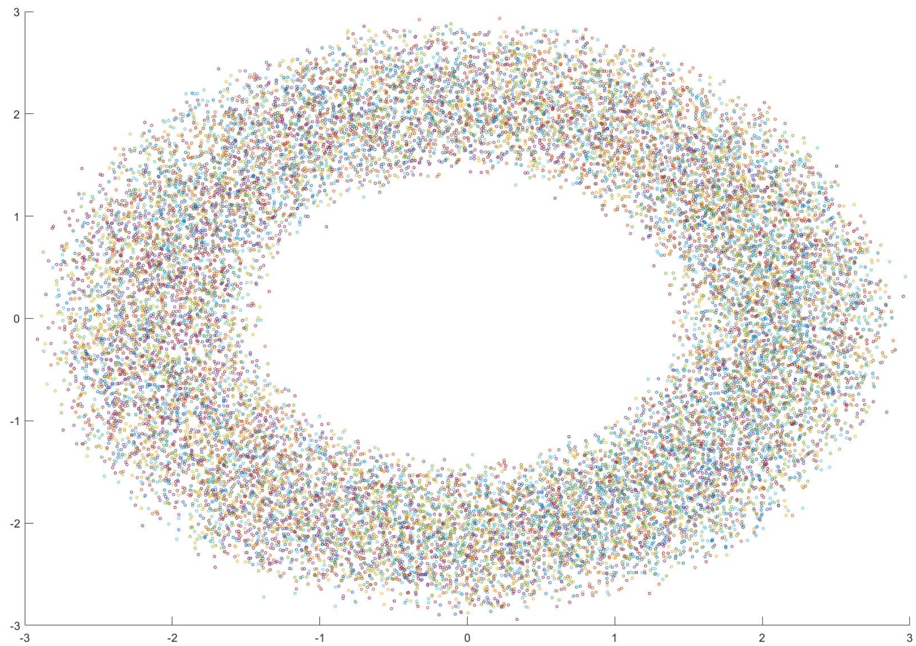


FIGURE 3.5: A simulation in 2 dimensions with N = 1000, λ = 2. Again, the colors represent the particle positions at stopping times, stopped every $10N$ births. In contrast to Figure 3.3, the particles have not spread out, and instead remain trapped in the annulus between $-\pi$ and π.

with $\Omega_{\ell(t)} = \{x \mid V(x) \geq \ell(t)\}$. A computational comparison of the solution to 3.29 to the particle solution was done in Python 3.5 using the Kolmogorov distance (i.e. the distance between the CDFs). At fixed times, we compared the CDF of $u(x,t)\,dx$ to the CDF of the empirical measure. Figure 3.6 shows a comparison of the CDFs at $T = 9$. The graph shows that the convergence of the CDFs is slow at the singularity of the PDE solution. Figure 3.7 shows the distance between the CDFs at $T = 9$ for various values of N. Because the measures are random, simulations were run 100 times for each N, and the average of the squared Kolmogorov distance for each N value was plotted. Values spanning $N = 100$ to 51200 were run. This free boundary PDE can explain both behaviors observed by the simulations. The trapped behavior is a result of the existence of a stationary solution for this PDE in certain parameter

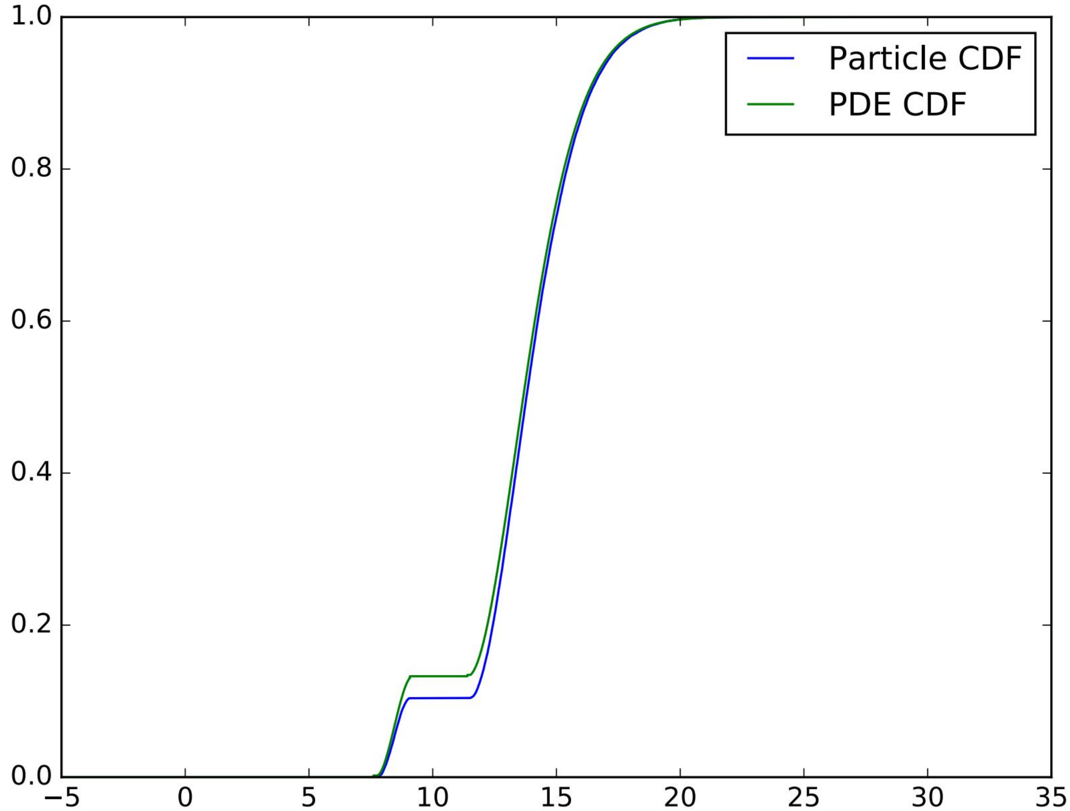


FIGURE 3.6: A comparison of the CDFs of the PDE solution and particle system empirical measure at $T = 9$ with $N = 51200$.

regimes. Let m_1 be a local minimum of $V(x)$ and $m_0 = \sup\{x < m_1 \mid V(x) = V(m_0)\}$; if $m_1 - m_0 > \frac{\pi}{\sqrt{\lambda}}$, then there exists an $a, b \in [m_0, m_1]$ such that $V(a) = V(b)$ and $b - a = \frac{\pi}{\sqrt{\lambda}}$. If such an a and b can be found, then the stationary solution is

$$\begin{aligned} u(x,t) &= \frac{\sqrt{\lambda}}{2} \sin\left(\sqrt{\lambda}(x-a)\right) \quad \text{for } x \in [a,b] \\ u(a,t) &= u(b,t) = 0 \end{aligned} \tag{3.30}$$

Figure 3.4 includes a plot of the conjectured stationary distribution compared to the density of the particles after 200,000 birth events. With $\lambda = 2$ and $V(x) = x + 2\sin(x)$, $a \approx 1.128$ and $b \approx 3.350$.

In higher dimensions, we still expect $\mu_t^N(dx) \to u(x,t)dx$ with $u(x,t)$ solving Equation 3.29. However, the precise form of the stationary solution is less clear, and

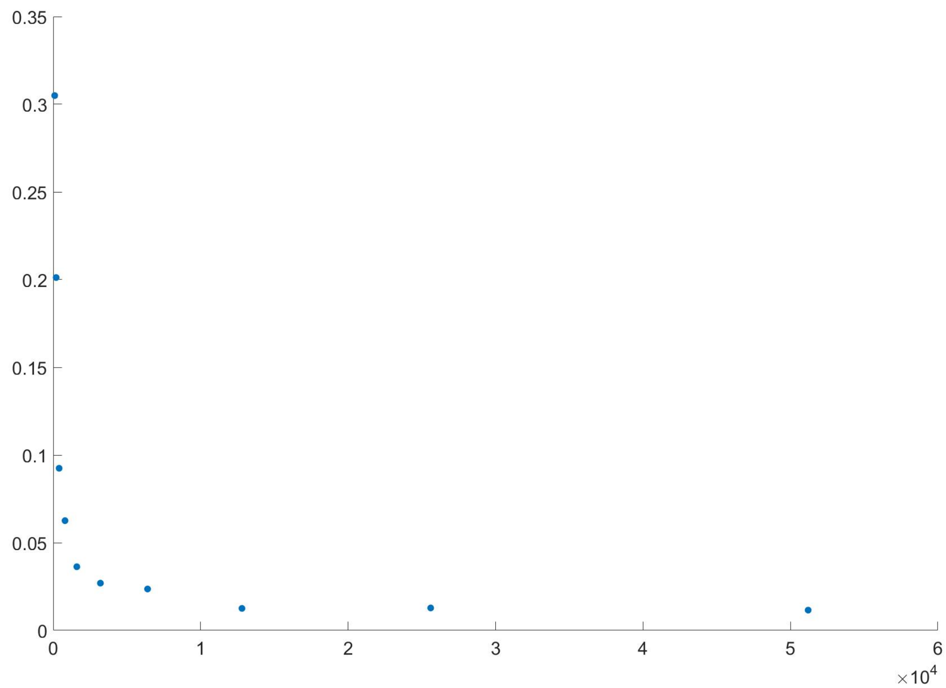


FIGURE 3.7: A plot of the Kolmogorov distance squared, averaged over 100 trials for each value of N. The Kolmogorov distance is between the particle system CDF and the free boundary PDE solution CDF at $T = 9$. The x-axis is the number of particles N, and the y-axis is the average of the distance squared.

smaller values of λ seem to be more conducive to the system moving, while higher values of λ result in the stuck behavior. Some values of λ, including $\lambda \approx .65$, give regular occurrences of both stuck and moving behavior with $N = 1000$.

4

N-BBM with a Symmetrically Decaying Fitness Function

Now we consider an N-BBM with a continuous fitness function which is symmetric, decays away from the origin, and has a unique local maximum at the origin. Such a fitness function removes the particle farthest away from the origin at each branch time. This system should no longer move with a positive speed, because the incentive is to remain as close to the origin as possible.

The motivation for studying this problem arose from the study of the process in Chapter 3. In particular, simulations showed that the population of particles can get stuck in a fitness peak for a long period of time, rather than traveling across $\mathbb{R}$. When clustered in a single peak, the particles appeared to spread out in the shape of a sinusoidal peak (see Figure 3.4). By removing all local peaks but one, we are able to study precisely the behavior of the particles while they remain stuck in a single peak. $V(x) = -x^2$ is an example of a fitness function in the framework we consider.

4.1 Relevant Previous Work

This problem is very similar to another well-known branching selection model, the Fleming-Viot model, which defines a fixed boundary and redistributes particles when they hit that boundary [9]. A precise version of the Fleming-Viot problem can be expressed as follows. Begin N independent Brownian motions in the interval $[a, b]$. Allow them to move freely until they hit the boundary, at which point they jump to the location of another particle in the system, chosen uniformly at random. It has been shown that if $b - a = \pi$, then as $N \to \infty$, the particles remain in a stationary distribution, with the (scaled) shape of $\sin(x)$ [9]. Studying the system with a free boundary (that is, one that moves with the maximum and minimum alive particle) requires different techniques than studying the process with the stationary boundary.

4.2 Main Results

In 2017, DeMasi et. al. [11] proved a hydrodynamic limit of N-BBM on $\mathbb{R}$ with monotonic fitness. In this section, we explain how to generalize their results to include an N-BBM system subject to a symmetric fitness function which monotonically decreases away from zero. One can consider $V(x) = -x^2$ as a concrete example of a function satisfying these conditions. Our main theorems are the corresponding hydrodynamic limit and the description of the limiting object in terms of a free boundary PDE.

To state our first theorem, we define μ_t^N to be the empirical measure of the system at time t:

$$\mu_t^N = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k(t)} \tag{4.1}$$

Let ρ be a function with $\rho \in L^1(\mathbb{R}, \mathbb{R}+)$ and ρ even and satisfying $||\rho||_\infty < \infty$ and $M_0 = \sup_r \left\{ \int_{-\infty}^{r} \rho(x)\, dx = 0 \right\} > -\infty$.

Theorem 12. *Let $X(t) \in \mathbb{R}^N$ be a system of N-BBMs on $\mathbb{R}$ with fitness function $V(x)$ which is symmetric about 0 and strictly decreases as $|x| \to \infty$. Choose each $X_k(0)$ independently and identically according to a density ρ, where ρ satisfies the conditions above. Then for every $t \geq 0$, there exists a probability density function $\Psi(x,t)$ such that for any $a \in \mathbb{R}$, we have*

$$\lim_{N\to\infty} \int_a^\infty \mu_t^N(dx) = \int_a^\infty \Psi(x,t)\, dx \tag{4.2}$$

almost surely and in L^1.

Theorem 13. *Suppose that $(u(\cdot,t), \ell(t))$ is a solution to the free boundary problem*

$$\begin{aligned} u_t &= u_{xx} + u \quad && -\ell \leq x \leq \ell, t > 0 \\ \int_{-\ell(t)}^{\ell(t)} u(x,t)\, dx &= 1 && \textit{for all } t \\ u(\ell(t),t) = u(-\ell(t),t) &= 0 && \textit{for all } t \\ u(0,x) &= \rho(x) \end{aligned} \tag{4.3}$$

on the time interval $[0,T]$ for some $T > 0$ with $\ell(t)$ continuous. Then the limiting function Ψ in Theorem 12 satisfies $\Psi(x,t) = u(x,t)$ for $t \in [0,T]$.

4.3 Generalization of DeMasi et. al. Results

We use the same notation as in the previous chapter, letting $X(t)$ represent the system of N particles at time t. Then define $R(0)$ to be the reflected initial configuration, with each particle strictly below 0. That is, $R_i(0) = -|X_i(0)|$ for all i. We couple $X(t)$ to $R(t)$, a system of reflecting BBMs by letting $R_k(t) = -|X_k(t)|$ for all particles k and all times t.

Notice that $R(t)$ has the same law as a system of N branching Brownian motions reflected at 0, with selection at the leftmost edge, where the initial location of particles is chosen according to the density $\tilde{\rho}$

$$\tilde{\rho}(x) = \begin{cases} 2\rho(x) & x \leq 0 \\ 0 & x > 0 \end{cases} \tag{4.4}$$

The particle selected at branch time τ in X is coupled to the particle at the leftmost edge of $R(\tau)$, because the selection choice only depends on the relative absolute value of the particles' positions, not the sign of the position. Therefore, the law of selection in the reflected system is selection according to the fitness function $V(x) = x$.

Similar to Chapter 2, we will choose to couple a system of reflecting BBMs subject to selection to a free system of reflecting BBMs, without any selection. Whenever we do this, we will do so by letting the coupled particles use the same Brownian increments and coupling the birth times. To be precise, suppose we want to couple a free system of branching Brownian motions, $S(t)$, to $R(t)$, each a system beginning with N reflecting BMs, but $R(t)$ having selection and $S(t)$ being free. Then for each $1 \leq i \leq N$, we have $R_i(t)$ defined by

$$dR_i(t) = dB_i(t) - dL_i^R(t)$$

for some Brownian motion $B_i(t)$ and a unique local time process $L_i^R(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}\{x - \varepsilon \leq B_i(s) + r_i \leq x + \varepsilon\}\, ds$, with $r_i = R_i(0) \in \mathbb{R}^-$. The coupled particle in $S(t)$ is defined by

$$dS_i(t) = dB_i(t) - dL_i^S(t)$$

with $B_i(t)$ the same Brownian motion used in the definition of $R_i(t)$, but with $S_i(0) = s_i \in \mathbb{R}^-$ not necessarily equal to r_i and therefore $L_i^R(t)$ not necessarily equal to $L_i^S(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}\{x - \varepsilon \leq B_i(s) + s_i \leq x + \varepsilon\}\, ds$. When selection occurs in $R(t)$, we mark the corresponding particle in $S(t)$ as removed, but allow it to continue to branch using an independent reflected BBM. At each branch time, particles must be relabeled appropriately to maintain the ordering. We use the labeling procedure of DeMasi et. al. and do not give a description here. See [11] for full details.

Using this method of coupling for reflected BM is important because it maintains the monotonicity of the system; that is, if $S_i(0) \geq R_i(0)$, then $S_i(t) \geq R_i(t)$ for all time (and vice versa) when S_i and R_i are coupled particles. If one system has fewer particles than the other, the coupling is the same, but there will be particles which are not coupled between the systems. If the free system has more particles, allow those particles to behave as independent reflected BBMs. If the selection systems has more particles, the extra particles will not be coupled to particles in $S(t)$.

In [11], the proof proceeds in the following way. First, the system is coupled to a upper bounding process and a lower bounding process. The upper and lower processes are obtained by strategically enforcing selection only at times $k\delta$, where the length of the time interval, δ, is fixed. They show the leftmost edge of the particles remains between the leftmost edge of each of the coupled bounding processes. Then it is shown that in the limit as $N \to \infty$, the position of the leftmost edges of these bounding processes converges to known upper and lower deterministic boundaries. Finally, they show that as $\delta \to 0$, the two deterministic boundaries converge to the same limit. Because of the coupling, we can say that the true hydrodynamic selection boundary also converges to that limit. Then all that remains is to show that the density converges to the solution of the desired free-boundary problem.

For the intermediate results, the proofs from [11] translate directly. This is because, as pointed out above, when we couple $X(t)$ to a system of negative reflected BBMs, the fitness function becomes $V(x) = x$, as it is in [11].

We define the following ordering between particle systems, which allows us to precisely define an upper and a lower bounding stochastic process.

If $X(t) = (X_1(t), X_2(t), \ldots, X_N(t))$ and $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_M(t))$ with $M \geq$

N, then we say that

$$X(t) \preceq Y(t) \quad \text{if and only if} \quad |X \cap [a, \infty)| \leq |Y(t) \cap [a, \infty)|$$

Stochastic Bounds We define the two bounding particle selection systems.

Upper Bound Process Fix $\delta > 0$. We will define an upper-bounding process $U^\delta(k\delta) = \{U_i^\delta(k\delta)\}$ iteratively for $k \in \mathbb{N}$.

Define $U^\delta(0) = R(0)$. Assume that $U^\delta((k-1)\delta)$ is defined. Let $S^\delta(t)$ be a free reflected BBM coupled to $R(t)$ with initial particles positions $U^\delta((k-1)\delta)$. At time $t = k\delta$, we select the rightmost particles in $S^\delta(\delta)$ to include in $U^\delta(k\delta)$, removing the rest. Therefore, there are only N particles in $U^\delta(k\delta)$ for all $k \in \mathbb{Z}$. That is,

$$U^\delta(k\delta) = \left\{ S_i^\delta(\delta^-) \;\middle|\; \#\{S_j^\delta \;\middle|\; S_j^\delta(\delta^-) > S_i^\delta(\delta^-))\} < N \right\} \tag{4.5}$$

where

$$U_j^\delta(k\delta^-) = \lim_{t \to k\delta^-} U_j^\delta(t) \tag{4.6}$$

To understand why we call this an upper-bounding process, suppose that $U^\delta((k-1)\delta) \succeq R((k-1)\delta)$. Between the selection events, the particles will remain ordered because two reflected BMs which are coupled through the Brownian increments do not intersect, except possibly at $x = 0$. If a particle in $R(t)$ jumps during the time increment, then it moves to become coupled with a different particle in $S^\delta(t-(k-1)\delta)$. This implies that the particle systems remain ordered between the $k\delta$ times. At $t = k\delta$, we choose the N most fit particles from $S^\delta(\delta^-)$ to keep. Therefore, since $S^\delta(t - k\delta^-) \succeq R_{k\delta^-}$, we have that $U^\delta(k\delta) \succeq R(k\delta)$. Through the iterative definition and the fact that $U^\delta(0) = R(0)$, we can see that $U^\delta(k\delta) \succeq R(k\delta)$ for all $k \in \mathbb{N}$.

Lower Bound Process We can also create a process which bounds $R(t)$ from below. We define $D^\delta(k\delta)$ in a similarly recursive manner. Let $D^\delta(0) = R(0)$. Again suppose that $D^\delta((k-1)\delta)$ is defined.

In the lower bounding process, selection for the time increment $((k-1)\delta, k\delta]$ happens at time $(k-1)\delta$, rather than at $k\delta$ as in the upper bounding process. Again we couple to $R(t)$ a free reflected BBM $S_\delta(t)$. To determine which particles are kept in $D^\delta(k\delta)$, we remove particles from left to right such that, after being allowed to evolve freely for time δ, the number of particles in $D^\delta(k\delta)$ is less than or equal to N. To define this precisely, let

$$L_{(k-1)\delta} = \min\left\{ a \in D^\delta((k-1)\delta) \;\middle|\; \sum_{i=1}^{|D^\delta((k-1)\delta)|} N^i_{k\delta} \mathbb{1}\{S_{\delta,i}(0) \geq a\} \leq N \right\}$$

where $N^i_{k\delta}$ is the number of offspring of $S_{\delta,i}(0)$ alive at time δ. With this barrier defined, we can define

$$D^\delta(k\delta) = \left\{ S_{\delta,i}(\delta) \;\middle|\; S_{\delta,i}(0) \geq L_{(k-1)\delta} \right\} \tag{4.7}$$

where we have abused notation slightly in the standard way by allowing $S_{\delta,i}(0)$ to be the position of the ancestor of the particle $S_{\delta,i}(\delta)$ if the particle is not alive at time 0. Because we remove particles at time $(k-1)\delta$ based on the behavior at $k\delta$, we remove entire families of particles, rather than a single particle at a time as we did in defining the upper bounding process. This means that D^δ can have fewer than N particles contained in it.

There are some technicalities to be aware of when defining the coupling at jump values for the lower bounding process. See [11] for a careful description of the label reassigning that must occur to maintain the appropriate ordering.

Deterministic Bounds To define the deterministic barriers, let the heat kernel for Neumann boundary conditions at $x = 0$ (the reflected heat kernel) be defined as

$$G_t u(x) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi t}} \left(\mathrm{e}^{-(x-y)^2/2t} - \mathrm{e}^{-(x+y)^2/2t} \right) u(y)\, dy \tag{4.8}$$

and define the cut operator to be

$$C_m u(x) = u(x) \mathbb{1} \left\{ \int_x^0 u(y)\, dy < m \right\} \qquad \text{for } x \le 0 \tag{4.9}$$

From there, define the upper deterministic barrier to be

$$D_0^{\delta,+} \tilde{\rho} = \tilde{\rho} \quad \text{and} \quad D_{k\delta}^{\delta,+} \tilde{\rho} = (C_1 \mathrm{e}^{\delta} G_\delta)^k \tilde{\rho} \tag{4.10}$$

and the lower deterministic barrier to be

$$D_0^{\delta,-} \tilde{\rho} = \tilde{\rho} \quad \text{and} \quad D_{k\delta}^{\delta,-} \tilde{\rho} = (\mathrm{e}^{\delta} G_\delta C_{e^{-\delta}})^k \tilde{\rho} \tag{4.11}$$

Notice that these behaviors mirror the selection behavior of the upper and lower stochastic bounds described above. That is, in the upper bound, the process is allowed to evolve and grow to size e^{δ}, then is cut back down to mass 1. This is repeated for each increment of time δ. For the lower bound, the process is first cut to size $\mathrm{e}^{-\delta}$, then allowed to evolve and grow to size 1 in an increment of time δ.

4.4 Proof of Theorems 12 and 13

Proof of Theorem 12. With this setup, the proofs in [11] translate directly. In particular, we can use [11, Theorem 1] to say that for every $t \ge 0$, there is a density function $\tilde{\Psi}(\cdot, t) : \mathbb{R} \to \mathbb{R}^+$ such that for any $a \in \mathbb{R}$,

$$\lim_{N\to\infty} \int_a^\infty \tilde{\mu}_t^N(dr) = \int_a^\infty \tilde{\Psi}(r,t)\, dr \qquad \text{a.s. and in } L^1 \tag{4.12}$$

where $\tilde{\mu}_t^N = \frac{1}{N} \sum_{k=1}^N \delta_{R_k(t)}$ is the empirical distribution of $R(t)$.

We also know from [11, Theorem 2], modified to accommodate reflection, that this $\tilde{\Psi}(\cdot,t)$ coincides with the solution to the free boundary PDE, $(\tilde{u}(\cdot,t),\ell(t))$ for $t \le T$,

$$\begin{aligned} \tilde{u}_t &= \frac{1}{2}\tilde{u}_{xx} + \tilde{u} \quad \text{for } x \in (\ell(t),0) \\ \tilde{u}(x,0) &= \tilde{\rho} \text{ for } x > L_0 \quad \text{and } \tilde{u}(x,0) = 0 \text{ for } x < L_0, \\ \tilde{u}(\ell(t),t) &= 0 \quad \text{and } \int_{\ell(t)}^0 \tilde{u}(x,t)\,dx = 1 \\ \tilde{u}_x(0,t) &= 0 \quad t > 0 \end{aligned} \tag{4.13}$$

where L_0 is defined as the largest value such that $\int_{L_0}^0 \tilde{\rho}(x)\,dx = 1$.

Now that have applied these theorems to the reflected version of the process, we must unfold the process back to $\mathbb{R}$ to make the necessary statements about μ_t^N.

We will show that $\mu_t^N(I^+) - \mu_t^N(I) \to 0$ for any interval $I \subset \mathbb{R}^-$ and corresponding positive interval $I+$, defined as the interval such that $x \in I^+$ if and only if $-x \in I$. This, combined with the application of the results from DeMasi et. al. [11] above will be enough to prove the desired result.

Given a realization $\mu_t^N(dx)$ of the process, we define the random variables $F_k(t)$, the families at time t. We say that two particles $X_i(t), X_j(t)$ are in the same family if and only if they have a common ancestor and neither X_i nor X_j has hit 0 since the time of the most recent common ancestor. There are at most N distinct families $F_k(t)$. See that if we let $\tilde{F}_k(t)$ be collection of particles in the underlying free BBM which are descendants of particle $X_k(0)$, then there is a function κ on $\{1,2,\ldots,N\}$ such that $|F_k(t)| \le |\tilde{F}_{\kappa(k)}(t)|$. That is, we can stochastically dominate the distribution of $\max_k |F_k(t)|$ by the distribution of $\max_{k=1,\ldots,N} H_k$, where each H_k is an independent $\mathrm{Geo}(\mathrm{e}^{-t})$ random variable. This gives us the following lemma.

Lemma 14. *Fix a time $T > 0$ and a constant $\alpha > 0$. Then there exists a constant*

C such that

$$\lim_{N\to\infty} \mathbb{P}\left(\max_k \frac{|F_k(T)|}{C\ln(N)} \geq \alpha\right) = 0$$

.

Proof. Let $\tilde{F}_k(T), \tilde{F}(T)$ be independent $\mathrm{Geo}(\mathrm{e}^{-T})$ random variables. Then

$$\begin{aligned}
\mathbb{P}(\max_k |F_k(T)| \leq \alpha C\ln(N)) &\geq \mathbb{P}(\max_k |\tilde{F}_k(T)| \leq \alpha C\ln(N)) \\
&= \prod_{k=1}^{N} \mathbb{P}(|\tilde{F}_k(T)| \leq \alpha C\ln(N)) \\
&= \left(\mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N))\right)^N
\end{aligned}$$

We want to show that this right hand side goes to 1 as $N \to \infty$. That is equivalent to showing that

$$N\ln\left(\mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N))\right) \to 0$$

as $N \to \infty$. Because $|\tilde{F}(T)| \stackrel{d}{=} \mathrm{Geo}(\mathrm{e}^{-T})$, we know that

$$1 - (1-\mathrm{e}^{-T})^{\alpha C\ln(N)} \leq \mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N)) \leq 1 - (1-\mathrm{e}^{-T})^{\alpha C\ln(N)+1}$$

Let $x = (1-\mathrm{e}^{-T})^{\alpha}$ and notice that $0 < x < 1$. Plugging in, we have

$$N\ln(1 - x^{C\ln(N)}) \leq N\ln\left(\mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N))\right) \leq N\ln(1-(1-\mathrm{e}^{-T})x^{C\ln(N)})$$

$$N(-x^{C\ln(N)})) \leq N\ln\left(\mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N))\right) \leq N(-(1-\mathrm{e}^{-T})x^{C\ln(N)}))$$

We can write $x = \mathrm{e}^{-b}$ for some b, which allows us to write $x^{C\ln(N)} = N^{-Cb}$. Pick $C = 3/b$ and we see that

$$-N \cdot N^{-3} \leq N\ln\left(\mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N))\right) \leq -N(1-\mathrm{e}^{-T})N^{-3}$$

$$\frac{-1}{N^2} \leq N\ln\left(\mathbb{P}(|\tilde{F}(T)| \leq \alpha C\ln(N))\right) \leq \frac{-(1-\mathrm{e}^{-T})}{N^2}$$

Taking $N \to \infty$, we get the desired result. $\square$

In fact, because the bound we proved decays to 0 so quickly, we can in fact apply Borel-Cantelli to say that $\limsup_{N\to\infty} \frac{1}{C\ln(N)} \max_k |F_k| < \alpha$ a.s. as well.

We now prove a lemma regarding the convergence of the different between the measure of I and I^+ a.s. and in L^1.

Lemma 15. *$\lim\limits_{N\to\infty} \mu_t^N(I^+) - \mu_t^N(I) = 0$ a.s and in L^1.*

Proof. First we show the L^1 convergence. Let $I \subset \mathbb{R}^-$ be an interval and I^+ be the reflection of that interval to $\mathbb{R}^+$. Define F_I as the number of families in interval I at time t, N_I as the number of particles in I in the reflected system at time t, and $M_N = \max_{k=1,\ldots,F_I} |F_k|$ to be the maximum family size in interval I at time t in the reflected system. Then we have that

$$\limsup_{N\to\infty} \frac{M_N}{C\ln(N_I)} < \alpha$$

almost surely for some C. This means that for any fixed $\varepsilon > 0$,

$$\mathbb{P}(M_m < \varepsilon C\ln(m_I) \text{ for all } m \geq N) \to 1$$

as $N \to \infty$ (while this may seem like the introduction of new random variables, M_m, m_i are just defined as the random variables M_N and N_I with $N = m$). Let ξ_k be i.i.d. uniform on $\{-1, 1\}$. Now define

$$Y_N = \frac{1}{N}\sum_{k=1}^{F_I} |F_k|\xi_k = \sum_{k=1}^{F_I} a_k\xi_k$$

which is equal in distribution to the difference between the measure of I^+ and I under μ_t^N, with $a_k = \frac{|F_k|}{N}$. We want to show that Y_N goes to 0 in L^1. We will in fact show that it goes to 0 in L^2.

We see that

$$\begin{aligned}
\mathbb{E}[Y_n^2] &= \mathbb{E}\left[\sum_{k=1}^{F_I} a_k^2 + \sum_{k\neq j} a_k a_j \xi_k \xi_j\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{F_I} a_k^2\right]
\end{aligned}$$

because ξ_k, ξ_j are independent.

Now suppose we define $\tilde{Y}_k$ to be the random variables obtained by merging families together until each group has at least $\varepsilon C \ln(N_I)$ members and has no more than $2\varepsilon C \ln(N_I)$ members; the last group may have less than $\varepsilon C \ln(N_I)$ particles if there are not enough particles left. Grouping in this way is possible if $M_n < \varepsilon C \ln(N)$ (an event which has probability 1 in the limit). Therefore, we can create the random variable $\tilde{Y}_n$ where

$$\tilde{Y}_n = \sum_{k=1}^{G_I} \tilde{a}_k \tilde{\xi}_k$$

where again each $\tilde{\xi}_k$ is chosen i.i.d. uniformly from $\{-1, 1\}$. Let B be the event that $M_N < \varepsilon C \ln(N)$. Then $\tilde{Y}_n$ has variance

$$\begin{aligned}
\mathbb{E}[\tilde{Y}_n^2] &= \mathbb{E}\left[\sum_{k=1}^{G_I} \tilde{a}_k^2 \;\Big|\; B\right] \mathbb{P}(B) + \mathbb{E}\left[\sum_{k=1}^{G_I} \tilde{a}_k^2 \;\Big|\; B^c\right] \mathbb{P}(B^c) \\
&\leq \mathbb{E}\left[\left(\frac{N_I}{\varepsilon C \ln(N_I)} + 1\right)\left(\frac{2\varepsilon C \ln(N_I)}{N}\right)^2\right] \mathbb{P}(B) + 1 \cdot \mathbb{P}(B^c) \\
&\leq \mathbb{E}\left[\frac{4\varepsilon C \ln(N_I)}{N_I} + \left(\frac{2\varepsilon C \ln(N_I)}{N}\right)^2\right] \mathbb{P}(B) + \mathbb{P}(B^c)
\end{aligned}$$

Notice that because of the almost sure bound we have on M_N, we know that $\mathbb{P}(B^c) \to 0$ as $N \to \infty$. So $\mathrm{Var}(\tilde{Y}_n) \to 0$ as $N \to \infty$. But in fact, $\tilde{a}_k = a_1^k + \cdots + a_{n(k)}^k$ for some

$n(k)$. But $(\tilde{a}_k)^2 \geq (a_1^k)^2 + \cdots + (a_{n(k)}^k)^2$ so

$$\mathrm{Var}(Y_n) \leq \mathrm{Var}(\tilde{Y}_n)$$

for all n. Therefore, $\mathbb{E}[Y_n^2] \to 0$ as $n \to \infty$ as well and Y_n converges in L^2 and so also converges in L^1 as desired.

Next, we want to show that $\mu_t^N(I^+) - \mu_t^N(I) \to 0$ a.s.

Consider an interval I with $I \subset \mathbb{R}^-$ and $\int_I \tilde{\Psi}(x,T)\,dx = C_I > 0$. Again we let N_I be the number of particles in I at time T. Clearly, this is a function of N and $N_I \to \infty$ a.s. as $N \to \infty$. Divide these N_I particles into groups of size $C\ln(N)$ without splitting up any families. In order to keep families together, we have to allow for small error; that is, the sizes can be $C\ln(N) + \varepsilon$ for $\varepsilon = o(\ln(N))$. We know that the number of families of size 1 grows at least like $\mathrm{e}^{-T}N_I$, and using these small families allows us to make these groups the appropriate size once N is sufficiently large. There will be G total groups with $G = O(N_I/\ln(N))$. To each of these groups, assign a random variable ξ_k which is -1 with probability $1/2$ and 1 with probability $1/2$. This variable will indicate whether the group belongs on the positive side of the axis or the negative side of the axis. After assigning each group an ξ_k, we can calculate $S_N = \frac{1}{N}\sum_{k=1}^{G}|G_k|\xi_k$, where $|G_k|$ is the size of group G_k. This random variable is the difference between the number of particles assigned to the right and the number of particles assigned to the left, divided by N.

As $N \to \infty$, we can see that

$$
\begin{aligned}
\lim_{N\to\infty} S_N &= \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{G} |G_k|\xi_k \\
&\leq \lim_{N\to\infty} \frac{1}{N}(C\ln(N)+\varepsilon)\sum_{k=1}^{G} \xi_k \\
&\leq \lim_{N\to\infty} \tilde{C}\frac{N_I}{N}\frac{1}{G}\sum_{k=1}^{G} \xi_k \\
&= 0 \quad \text{a.s. and } \mathrm{L}^1
\end{aligned}
$$

by the law of large numbers (since $G \to \infty$ as $N \to \infty$ and the ξ_k are all independent with $\mathbb{E}[\xi_k] = 0$).

This is not quite enough however. Notice that $S \neq \mu_t^N(I^+) - \mu_t^N(I)$, because each family was not assigned a side independently. However we make the following claim:

Claim: Any additional independent assignments will make make S_N closer to 0 with probability $p > \frac{1}{2}$.

That is, suppose we pick a particle uniformly at random from the N_I particles. Suppose the selected particle is in group G_k. If the number of families in G_k is at least 2, then we pick a family in G_k and assign it a new $\xi_{k'}$ and separate it into its own group. This action has a probability $p > 1/2$ of making the new S'_N closer to 0. To see this, suppose $S_N > 0$. There are more particles assigned a $+1$ than particles assigned a -1. Therefore, we are more likely to select a family which is assigned an $\xi_k = 1$. Reassigning a family in that group will either keep S_N the same or increase the number of -1 assignments by reassigning a family to -1. Therefore, this will move S'_N closer to 0 than S_N. If a group assigned a -1 is selected, then S'_N will either equal S_N or be farther from 0. Because choosing a positive is more likely, there is a greater probability of moving the sum towards 0 with each additional assignment than away from it.

We can repeat this process until all families are assigned an independent ξ_k. Because the number of families goes to infinity as $N \to \infty$, the number of reassignments that are needed will get large as N gets large. Therefore, the probability that the true sum $\mu_t^N(I^+) - \mu_t^N(I)$ is closer to 0 than S_N approaches 1 as $N \to \infty$.
This allows us to say that $\mu_t^N(I^+) - \mu_t^N(I) \to 0$ as $N \to \infty$ a.s. $\square$

Therefore, since $\mu_t^N(I^+) + \mu_t^N(I) \to C_I$ a.s. and in L^1 and the previous lemma tells us that $\mu_t^N(I^+) - \mu_t^N(I) \to 0$ a.s. and in L^1, we know that $\mu_t^N(I) \to \frac{C_I}{2}$ a.s. and in L^1. This means that the mass in any interval I is evenly split between the positive and the negative axes. This gives us the desired result: if

$$\Psi(x,t) = \begin{cases} \frac{\tilde{\Psi}(x,t)}{2} & x \leq 0 \\ \frac{\tilde{\Psi}(-x,t)}{2} & x > 0 \end{cases}$$

then

$$\lim_{N\to\infty} \int_{-\infty}^{a} \mu_t^N(dx) = \int_{a}^{\infty} \Psi(x,t)\, dx$$

almost surely and in L^1 for all $a \in \mathbb{R}$. This concludes the proof of Theorem 12. $\square$

The proof of Theorem 13 is essentially one line.

Proof of Theorem 13. All that remains to prove this theorem is to say that because $\tilde{\Psi}$ solves the reflected free boundary problem, wherever such a solution exists, then $\Psi(x,t)$ solves the desired free boundary problem on all of $\mathbb{R}$. This is clearly true because the initial conditions were chosen to be symmetric and the heat equation is symmetric, so $\Psi(x,t)$ matches the solution to Equation 4.3 wherever one exists. $\square$

5

BBM with Branch Rate Interaction via the Empirical Measure

5.1 Introduction

In this chapter, we study a BBM where the particle interaction occurs through the branch rate of each particle. We allow the branch rate of each individual to be a function of the empirical measure of the process, smoothed through convolution. Defining a branch rate which depends on the empirical measure creates a nonlocal effect; particles are affected by the entire configuration. We say that the interaction between particles is *weak* because the influence of a single particle is on the order $1/N$, where N is the number of initial particles.

5.1.1 Related Work

McKean-Vlasov SDEs Allowing particle systems to interact through the system's empirical measure is well-established. Historically, the interaction has been included in the drift and diffusion terms in an interacting system of SDEs. In this case, taking the hydrodynamic limit results in a solution to a McKean-Vlasov stochastic differential equation. This problem and its generalizations have been studied extensively

(see for instance [19, 20]). In 1989, Oelschläger looked at a multitype branching processes in which the dependence on the empirical measure is present in the branch rate of the particles [21]. He considers the system with moderate interaction strength, where the birth, death, and type transition rates of a single particle depend upon the other particles in a neighborhood whose volume goes to 0 as N goes to infinity, but which contains infinitely many particles as N goes to infinity. This is in contrast to a weak interaction, where the neighborhood of interaction has a constant size. He proves a hydrodynamic limit of the system to a general system of reaction-diffusion equations. He claims, but does not show, that his results can be extended to the case of long-range interactions between particles (i.e. weak interaction).
In this work, we show the corresponding theorem for weakly interacting branching Brownian motions; however, we restrict to one type, rather than a multitype system. We will show this for a class of rate functions, described in Equation 5.2 below.

Discrete Process with Mean-Dependent Rate The motivation for this problem, however, came from a different perspective. In particular, while the work done below is for a positive, bounded, and Lipschitz rate function Λ, our interest in the problem stems from one particular rate function which corresponds to the following behavior: particles in front of the average position, call it M_t, branch at an instantaneous rate $X_k(t) - M_t$, while particles behind the average position die at a rate $M_t - X_k(t)$. This model can be thought of as a continuous version of a model introduced by Yu, Etheridge, and Cuthbertson [23]. In this paper, Yu, Etheridge, and Cuthbertson use a Moran model to study populations whose fitness is evolving on $\mathbb{Z}$. A particle's fitness can change via mutation (to the fitness above or below their current level), via selection (where individuals of higher fitness replace those of lower fitness), and through resampling (where an individual replaces another at a constant rate regardless of fitness). The discrete setup avoids many of the difficulties which arise in the continuous time version. In particular, the fitness of the particles does not continually evolve and the birth and death events are coupled together in the selection and resampling events. Therefore, the number of particles at any time is fixed.

5.1.2 Contribution and Difficulties

Despite being motivated by the study of a system where the birth and death rates are proportional to the distance from the mean, the proofs below do not encompass the choice of Λ, Φ that is required in this case; we cannot yet prove the result for the motivating example. While we extend the discrete model to the case of particles whose fitness evolves continuously and allow for a changing population size, we have restricted our study to a bounded, positive rate function. We would need to allow the rate to be negative and unbounded in order to make statements about a system in which the birth and death rates are proportional to the particle's distance from the mean. There are several considerations to take into account before extending to a negative or unbounded rate function.

Incorporating death rates Incorporating particles dying can be done by allowing the branch rate to be negative and interpreting negative branch rates as death rates. In this case, one must consider the possibility of the system becoming degenerate at some time. In the case described above where the birth/death rates are proportional to a particle's distance from the mean, the system will reach a state with only one particle remaining with probability one. This particle will then continue as an independent Brownian motion and no longer branch. One approach to overcoming this difficulty could be to make the claim that the probability of reaching this state before a fixed time T should go to 0 as N goes to infinity. After verifying this, the proof of proving the hydrodynamic limit on the time interval $[0, T]$ would be similar to the proof presented here.

Incorporating unbounded rate functions Unbounded rate functions present more of a problem than allowing particles to die. At several places in the proof of the theorem, we rely on control of the number of particles alive at time t. This control is obtained by using the maximum branch rate as an upper bound on the true branch rate. This technical difficulty perhaps less daunting than the second difficulty introduced by unbounded rates. Such a generalization also introduces the possibility of finite time explosion of the number of particles (see [5] for one treatment of finite time explosion in a noninteracting system of BBMs). Ensuring that there is no finite time explosion of a system with unbounded branch rates would require substantially more delicate analysis of the large deviation events of the process.

5.2 Formal Problem Statement

Let $X_1, \ldots, X_N$ be a collection of N binary BBMs. Allow their initial positions to be independently distributed with density ρ. We let A_t be the collection of the indicies of particles alive at time t and define

$$\mu_t^N = \frac{1}{N} \sum_{u \in A_t} \delta_{X_u(t)} \tag{5.1}$$

to be the empirical measure of the system at time t.

Particle $X_u(t)$ splits into two at an instantaneous rate $\lambda(X_u(t), \mu_t^N)$, defined as

$$\lambda(x, [\mu]) = \Lambda \left(\int_{\mathbb{R}} \Phi(x - y) \mu(dy) \right) \tag{5.2}$$

for positive, bounded, and Lipschitz $\Lambda : \mathbb{R} \to \mathbb{R}$ and Φ a smooth, compact function. Notice that λ is continuous in both x and μ (see Appendix A for these details). For notational compactness, $\lambda(x, [\mu_t^N])$ will be written $\lambda(x)$ unless we need to explicitly draw attention to the measure dependence, and we will use $\star$ to represent convolution.

Notice that, unlike previous chapters, the index of a particle does not indicate anything about its position relative to other particles. As such, we will use the Ulam-Harris notation in this chapter to refer to the index of particles.

5.3 Main Result

Our main result is the associated hydrodynamic limit and the description of the limiting object in terms of a solution to a reaction-diffusion equation. We make the following hypothesis:

Hypothesis 1: For some time $T > 0$

$$\begin{aligned} u_t &= \Delta u + \Lambda(\Phi \star u)u \quad && x \in \mathbb{R}, t > 0 \\ u(x,0) &= \rho(x) \quad && x \in \mathbb{R}, t = 0 \end{aligned} \tag{5.3}$$

has a unique $C_b^\infty([0,T] \times \mathbb{R}, \mathbb{R})$ solution.

Theorem 16. *Let X be a collection of N binary BBMs with rate λ as defined in Equation 5.2 and the initial positions of each BBM chosen i.i.d. according to a density $\rho(x)$. Suppose Hypothesis 1 is true. Then the empirical measure of X has a weak limit in $D([0,T], \mathcal{M}_+(\mathbb{R}))$:*

$$\lim_{N\to\infty} \mu_t^N(dx) = \mu_t(dx) \tag{5.4}$$

Additionally, if $u(x,t)$ is the unique solution on $[0,T]$ to the equation

$$\begin{aligned} u_t &= \Delta u + \Lambda(\Phi \star u)u \quad && x \in \mathbb{R}^d, t > 0 \\ u(x,0) &= \rho(x) \quad && x \in \mathbb{R}^d, t = 0 \end{aligned} \tag{5.5}$$

Then $\mu_t(dx) = u(x,t)\,dx$ on this time interval.

5.4 Proof of Hydrodynamic Limit

The proof of the hydrodynamic limit in Theorem 16 proceeds in a fairly standard way. First, tightness of the sequence of probability measures is proven in the space $D([0,T],\mathcal{M}_+)$. Then the limit of a convergent subsequence is identified as the weak solution to the PDE. Last, uniqueness of the subsequence limits follows from the uniqueness of solutions to the PDE and then applied to give an overall limit of the empirical measures. An example of this in the simpler case of BBM is shown in Appendix A.

5.4.1 Tightness of the measure-valued processes

Let ν_N be the law of μ_t^N on the space of functions $D([0,T],\mathcal{M}_+)$, where $\mathcal{M}_+$ is the space of positive, finite measures on $\mathbb{R}$. Also define $C_0(\mathbb{R},\mathbb{R})$ as the space of continuous functions which decay to 0 at ∞ and $-\infty$. For a function $f \in C_0(\mathbb{R},\mathbb{R})$, we define $\pi_f : D([0,T],\mathcal{M}_+) \to D([0,T],\mathbb{R})$ in the following way

$$\pi_f \mu = \int_{\mathbb{R}} f(x)\,\mu(dx) \tag{5.6}$$

We will prove tightness in three steps. First, we will show that for $f \in C_0$, $\{\pi_f \mu_t^N\}$ satisfies the Aldous condition, defined below. Then we will show that in fact this collection $\{\pi_f \mu_t^N\}_N$ is tight in $D([0,T],\mathbb{R})$. Finally, we will apply a theorem of Roelly-Coppoletta which says that this is enough to get tightness in the space $D([0,T],\mathcal{M}_+)$.

5.4.2 Aldous Condition

The Aldous condition can be stated as follows [16]. Let Y_n be a real-valued process.

Definition 17. *[Aldous Condition] For all $\varepsilon > 0, \eta > 0$ there exists a $\delta > 0$ and an integer n_0 such that for any family of stopping times $\{\tau_n\}_n$ with $\tau_n \leq T$*

$$\sup_{n \geq n_0} \sup_{\theta \leq \delta} \mathbb{P}^n \left(|Y_n(\tau_n + \theta) - Y_n(\tau_n) \geq \eta \right) \leq \varepsilon \tag{5.7}$$

Lemma 18. *Let $f \in C_0$. The processes $Y_N = \pi_f \mu_t^N$ satisfy the Aldous condition.*

Proof. Fix $\varepsilon, \eta > 0$ and a collection of stopping times $\{\tau_n\}$ with $\tau_n \leq T$. We first pick an a such that $\mathbb{P}(N_{\tau_n} > N(\mathrm{e}^{M_\Lambda T} + a)) \leq \frac{\varepsilon}{3}$ for each τ_n, with $M_\Lambda = ||\Lambda||_\infty$ the maximum of the rate function. It is clear that because each of the particles X_k is branching with a rate bounded by M_Λ, $N_t \preceq \sum_{k=1}^{N} G_k$, where the G_k's are independent and $G_k \stackrel{d}{=} \mathrm{Geo}(\mathrm{e}^{-M_\Lambda t})$. Noticing that the variance of this sum is $O(N)$, we apply Chebyshev's inequality:

$$\begin{aligned}
\mathbb{P}(N_{\tau_n} \geq N(\mathrm{e}^{M_\Lambda T} + a)) &\leq \mathbb{P}\left(\left|\sum_{k=1}^{N} G_k - N\mathrm{e}^{M_\Lambda T}\right| \geq aN\right) \\
&\leq \frac{\mathrm{e}^{2M_\Lambda t}(1 - \mathrm{e}^{-M_\Lambda t})}{Na^2} \qquad (5.8)\\
&\leq \frac{\mathrm{e}^{2M_\Lambda T}}{Na^2}
\end{aligned}$$

Therefore, we can choose an a large enough such that

$$\frac{\mathrm{e}^{2M_\Lambda T}}{Na^2} < \frac{\varepsilon}{3}$$

which implies that

$$\mathbb{P}(N_{\tau_n} \geq NM_N) < \frac{\varepsilon}{3}$$

where we have defined $M_N = \mathrm{e}^{M_\Lambda T} + a$. Therefore, we can focus on bounding

$$\mathbb{P}(|Y_N(t+\theta) - Y_N(t)| \geq \eta \;\Big|\; N_t < NM_N)$$

for any time t.

Expanding this difference slightly, we see that

$$
\begin{aligned}
|Y_N(t+\theta) - Y_N(t)| &= \frac{1}{N}\left|\sum_{u\in A_t} f(X_u(t+\theta)) - f(X_u(t)) + \sum_{u\in A_{t+\theta}\setminus A_t} f(X_u(t+\theta))\right| \\
&\le \frac{1}{N}\sum_{u\in A_t} |f(X_u(t+\theta)) - f(X_u(t))| + \sum_{u\in A_{t+\theta}\setminus A_t} |f(X_u(t+\theta))|
\end{aligned} \tag{5.9}
$$

All of the particles in the first sum are independent in the time interval t to $t+\theta$ because any particles born during that time interval are addressed in the second sum. Using the same argument we used to bound N_t, we can pick a δ_1 small enough and n_0 large enough that

$$
\mathbb{P}\left(N_{t+\delta_1} - N_t \ge \frac{N}{M_f}\frac{\eta}{2} \;\middle|\; N_t < NM_N\right) < \frac{\varepsilon}{3} \tag{5.10}
$$

for all $N \ge n_0$, with $M_f = ||f||_\infty$. Using this bound, for $N \ge n_0$ and $\theta \le \delta_1$, we see that

$$
\begin{aligned}
\frac{1}{N}\sum_{u\in A_{t+\theta}\setminus A_t} |f(X_u(t+\theta))| &\le \frac{1}{N}(N_{t+\delta_1} - N_t)M_f \\
&< \frac{1}{N}\frac{N}{M_f}\frac{\eta}{2}M_f \\
&= \frac{\eta}{2}
\end{aligned} \tag{5.11}
$$

Because $f \in C_0$, for each $u \in A_t$ there exists a Δx_u such that if $|X_u(t) - y| \le \Delta x_u$, then $|f(X_u(t)) - f(y)| \le \frac{\eta}{2M_N}$. Define $\Delta x = \min_{u\in A_t} \Delta x_u$ and pick δ_2 small enough such that if we define

$$
s = \mathbb{P}\left(\sup_{0\le s\le \delta_2} |B(s)| > \Delta x\right)
$$

for $B(s)$ a BM started at 0, then s is small enough that

$$
sNM_N < \frac{\varepsilon}{3}
$$

This choice ensures that if E is the event that all particles alive at time t remain within Δx of their starting position in the time interval $(t, t+\delta_2)$, then

$$\begin{aligned}
\mathbb{P}\left(E^c \,\middle|\, N_t \leq NM_N\right) &\leq \sum_{u \in A_t} \mathbb{P}(X_u \text{ leaves a } \Delta x \text{ interval}) \\
&\leq sNM_N \\
&< \frac{\varepsilon}{3}
\end{aligned}$$

If all N_t particles remain within Δx of their initial position during the time interval, then

$$\begin{aligned}
\frac{1}{N} \sum_{u \in A_t} |f(X_u(t+\theta)) - f(X_u(t))| &< \frac{1}{N} NM_N \frac{\eta}{2M_N} \\
&= \frac{\eta}{2}
\end{aligned}$$

Choose $\delta = \min(\delta_1, \delta_2)$. The inequalities above show that for $\theta < \delta$ and $N \geq n_0$, if $N_t < NM_N$, $N_{t+\theta} - N_t < \frac{N}{M_f}\frac{\eta}{2}$ and each of the N_t particles stay within Δx during the time interval $(t, t+\theta)$, then

$$|Y_N(t+\theta) - Y_N(t)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta \tag{5.12}$$

Therefore, the only way that $|Y_n(t+\theta) - Y_n(t)| \geq \eta$ is if one of these conditions fails. But by making each of these events sufficiently unlikely, we have ensured that

$$\begin{aligned}
\mathbb{P}\left(\{N_t \geq NM_N\} \bigcup \left\{N_{t+\theta} - N_t \geq \frac{N\eta}{2M_f}\right\} \bigcup E^c\right) &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned} \tag{5.13}$$

This shows that $\{Y_N\}$ satisfies the Aldous condition. □

Now, we will use the fact that the processes Y_N satisfy the Aldous condition to show that the laws of these processes are tight in $D([0,T], \mathbb{R})$.

Lemma 19. *Let $f \in C_0$. Then $\{\pi_f \mu_t^N\}_N$ is tight in $D([0,T],\mathbb{R})$.*

Proof. From [16], we know that tightness of the processes $Y_N = \pi_f \mu_t^N$ follows from two things: that the processes satisfy the Aldous condition and that for a dense subset of times t, $\{Y_N(t)\}_N$ is tight in $\mathbb{R}$. We already know that the processes satisfy the Aldous condition by Lemma 18.

To satisfy the other condition required for tightness in [16], we need to show that for a fixed t, the $\{Y_N(t)\}_N$ are tight. This must hold for a dense subset of times in $[0,T]$, but in fact we will show it for every value of t. Fix a time t and an $\varepsilon > 0$. We are looking for a compact set $A \subset \mathbb{R}$ such that

$$\mathbb{P}(Y_N(t) \in A^c) \leq \varepsilon$$

As before, we pick a large enough that

$$\mathbb{P}(N_t \geq N(\mathrm{e}^{M_\Lambda t} + a)) \leq \varepsilon$$

Define $M_N = \mathrm{e}^{M_\Lambda t} + a$ and pick

$$A = [\min(M_N m_f, 0), M_N M_f]$$

where $M_f = \max_x f(x)$ and $m_f = \min_x f(x)$. It is clear that if $N_t \leq N M_N$, then

$$\begin{aligned} Y_N(t) &= \frac{1}{N} \sum_{u \in A_t} f(X_u(t)) \\ &\leq \frac{1}{N} N_t M_f \\ &\leq \frac{1}{N} N M_N M_f \\ &= M_N M_f \end{aligned}$$

If $f \geq 0$, then it is clear that $Y_N(t) \geq 0$. If $f(x) < 0$ for some x, then we see that

$$\begin{aligned} Y_N(t) &= \frac{1}{N} \sum_{u \in A_t} f(X_u(t)) \\ &\geq \frac{1}{N} N_t m_f \\ &\geq \frac{1}{N} N M_N m_f \\ &= M_N m_f \end{aligned}$$

So the $\mathbb{P}(Y_N(t) \in A^c) \leq \mathbb{P}(N_t \geq N M_N) \leq \varepsilon$. Therefore, at each time t, the law on $\mathbb{R}$ is tight.

This guarantees that the processes $\{Y_N\}$ are tight in $D([0,T], \mathbb{R})$. $\square$

Finally, we are ready to make a statement about the tightness of the measure-valued processes.

Proposition 20. *The measure-valued processes $\{\mu_t^N\}_N$ are tight in the space $D([0,T], \mathcal{M}_+)$.*

Proof. By Lemma 19, we know that the processes $\pi_f \mu_t^N$ are tight in $D([0,T], \mathbb{R})$ for any $f \in C_0$. This means that for a dense collection $\{f_k\} \subset C_0$, we can say that the collection $\{\pi_{f_k} \mu_t^N\}_N$ is tight in $D([0,T], \mathbb{R})$. We apply Theorem 2.1 in [22] to state that this tightness implies tightness of $\{\mu_t^N\}_N$ in the space $D([0,T], \mathcal{M}_+)$ as desired. $\square$

5.4.3 Characterization of Limit Object

By Prokhorov's Theorem (see Appendix A for a precise statement), tightness of the processes μ_t^N is equivalent to being pre-compact in the weak topology; that means that every sequence has a convergent subsequence. We want to describe these limit objects by how they act on test functions. If the limit is the same for every convergent subsequence, then in fact the entire sequence $\{\mu_t^N\}$ converges to that limit.

We now consider μ_t^N applied to a test function.

Let $\phi \in C_c^\infty([0,T] \times \mathbb{R}, \mathbb{R})$ be our test function, A_t be the set of particles alive at time t, and $N_t = |A_t|$, the number of particles alive at time t. Also define τ_u to be the birth time of particle X_u. Let us abuse notation slightly by letting $\{\mu_t^N\}_N$ be some convergent subsequence, rather than the more tedious notation $\{\mu_t^{N_k}\}$. Then we have

$$N\left(\int_{\mathbb{R}} \phi(t,x)\mu_t^N(dx) - \int_{\mathbb{R}} \phi(0,x)\mu_0^N(dx)\right) = \sum_{u \in A_t} \phi(t, X_u(t)) - \sum_{u \in A_0} \phi(0, X_u(0)) \tag{5.14}$$

By Itô's formula, we get that this equals

$$\begin{aligned}
&= \sum_{u \in A_t} \phi(t, X_u(t)) - \phi(\tau_u, X_u(\tau_u)) + \sum_{u \in A_t} \phi(\tau_u, X_u(\tau_u)) - \sum_{u \in A_0} \phi(0, X_u(0)) \\
&\quad = \sum_{u \in A_t} \int_{\tau_u}^t \phi_t(s, X_u(s)) + \frac{1}{2}\phi_{xx}(s, X_u(s))\, ds + \int_{\tau_u}^t \phi_x(s, X_u(s))\, dX_u(s) \\
&\qquad\qquad + \sum_{u \in A_t} \phi(\tau_u, X_u(\tau_u)) - \sum_{u \in A_0} \phi(0, X_u(0)) \\
&\qquad\quad = \sum_{u \in A_t} \int_0^t (\phi_t + \frac{1}{2}\phi_{xx})\mathbb{1}\{\tau_u < s\}\, ds + M_t + \sum_{\substack{u \in A_t \\ \tau_u > 0}} \phi(\tau_u, X_u(\tau_u))
\end{aligned} \tag{5.15}$$

with $M_t = \sum_{u \in A_t} \int_{\tau_u}^t \phi_x\, dX_u(s)$ a martingale. We can rewrite the sums as integration against singular measures in the following way

$$\int_{\mathbb{R}} \phi(t,x)\mu_t^N(dx) - \int_{\mathbb{R}} \phi(0,x)\mu_0^N(dx) = \int_0^t \int_{\mathbb{R}} (\phi_t + \tfrac{1}{2}\phi_{xx})\, d\mu_s^N\, ds + \int_0^t \int_{\mathbb{R}} \phi\, d\eta^N(x,s) + \tfrac{M_t}{N} \tag{5.16}$$

with $d\eta^N(x,s)$ a singular measure on $(0,t] \times \mathbb{R}$.

We want to be able to take a limit of both sides to be able to show that μ_t, the limit of this subsequence, satisfies the weak formulation of the PDE in Theorem 16. That is, we hope to show that the limiting object μ_t satisfies the equation

$$\langle \phi, \mu_s \rangle|_0^t = \int_0^t \int_{\mathbb{R}} (\phi_t + \frac{1}{2}\phi_{xx})\, d\mu_s\, ds + \int_0^t \int_{\mathbb{R}} \phi(s,x)\Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu_s(dx)\right)\, \mu_s(dx)\, ds \tag{5.17}$$

where $\langle \phi, \mu_s^N \rangle = \int_{\mathbb{R}} \phi(x,s)\, \mu_s^N(dx)$. We will deal with the limit of the terms in Equation 5.16 separately. Note that eventually we will want to show weak convergence of 5.16. However, some of the limits will be shown in L^2 instead, which implies weak convergence. The first term on the right which we address is $\dfrac{M_t}{N}$.

Lemma 21.

$$\lim_{N\to\infty} \frac{M_t}{N} = 0 \quad \textit{in } L^2 \textit{ for all } t.$$

Proof. Consider $\mathrm{Var}\left(\frac{M_t}{N}\right) = \frac{1}{N^2}\mathbb{E}[M_t^2]$.

$$\begin{aligned}
\mathbb{E}[M_t^2] &= \mathbb{E}\Bigg[\sum_{u\in A_t}\left(\int_{\tau_u}^t \phi_x(s, X_u(s))\, dX_u(s)\right)^2 \\
&\quad + \sum_{\substack{u,v\in A_t\\ u\neq v}} \left(\int_{\tau_v}^t \phi_x(s, X_v(s))\, dX_v(s)\right)\left(\int_{\tau_u}^t \phi_x(s, X_u(s))\, dX_u(s)\right)\Bigg] \\
&= \mathbb{E}\left[\sum_{u\in A_t}\left(\int_{\tau_u}^t \phi_x(s, X_u(s))\, dX_u(s)\right)^2\right] \\
&\quad + \mathbb{E}\left[\sum_{\substack{u,v\in A_t\\ u\neq v}} \left(\int_{\tau_v}^t \phi_x(s, X_v(s))\, dX_v(s)\right)\left(\int_{\tau_u}^t \phi_x(s, X_u(s))\, dX_u(s)\right)\right] \\
&= (1) + (2)
\end{aligned}$$

Consider term (2) first. We are summing over pairs of particles alive at time t. Clearly particles from different lineages are independent, so those expectations can be multiplied and contribute nothing to the sum.

For particles X_u, X_v from the same lineage, we note that because we are only summing over a particle from its birth time until t, the only dependence between these particles is their starting point. Because they have independent increments, their Itô integrals are independent. Therefore, we once again multiply to get that the expectation is 0. So $(2) = 0$.

Now we focus on finding (1). Because each of the N initial lineages begin distributed according to ρ, they are identically distributed, so we can do the following:

$$\begin{aligned}\mathbb{E}[M_t^2] &= \mathbb{E}\left[\sum_{u\in A_t}\left(\int_{\tau_u}^t \phi_x(X_u(s))\,dX_u(s)\right)^2\right]\\ &= N\mathbb{E}\left[\sum_{u\in F_1(t)}\left(\int_{\tau_k}^t \phi_x(X_k(s))\,dX_k(s)\right)^2\right]\end{aligned}$$

where $F_1(t) = \{u \in A_t \bigm| (1) < u\}$. We can change these integrals to have the bounds 0 to t, by a slight abuse of notation which allows $X_u(s)$ to refer to the ancestor of $X_u(t)$ alive at time $s < \tau_u$.

$$\begin{aligned}\mathbb{E}[M_t^2] &\leq N\mathbb{E}\left[\sum_{u\in F_1(t)}\left(\int_{\tau_u}^t |\phi_x(X_u(s))|\,dX_u(s)\right)^2\right]\\ &\leq N\mathbb{E}\left[\sum_{u\in F_1(t)}\left(\int_0^t |\phi_x(X_u(s))|\,dX_u(s)\right)^2\right]\\ &= N\mathbb{E}\big[|F_1(t)|\big]\ \mathbb{E}\left[\left(\int_0^t |\phi_x(B(s))|\,dB(s)\right)^2\right]\end{aligned}$$

where the last equality is by the many-to-one lemma. Let $M_\Lambda = ||\Lambda||_\infty$ and $M_{\phi_x} =$

$||\phi_x||_\infty$.

$$\begin{aligned}\mathbb{E}[M_t^2] &\leq N\mathrm{e}^{M_\Lambda t}\mathbb{E}\left[\left(\int_0^t |\phi_x(B(s))|\, dB(s)\right)^2\right] \\ &= N\mathrm{e}^{M_\Lambda t}\mathbb{E}\left[\int_0^t \phi_x^2(B(s))\, ds\right] \\ &\leq N\mathrm{e}^{M_\Lambda t} M_{\phi_x}^2 t\end{aligned}$$

where we have used Itô's isometry to get the equality in the second line. Therefore,

$$\mathrm{Var}\left(\frac{M_t}{N}\right) \leq \frac{\mathrm{e}^{M_\Lambda t} M_{\phi_x}^2 t}{N} \to 0$$

as $N \to \infty$.

Therefore,

$$\lim_{N\to\infty} \frac{M_t}{N} = 0 \quad \text{in } L^2$$

for all t. □

5.4.4 Point Process Martingale

The next term we consider is the integral against the point process η^N, from Equation 5.16. Recall that

$$\int_0^t \int_{\mathbb{R}} \phi\, d\eta^N(x,s) = \frac{1}{N} \sum_{\substack{u\in A_t \\ \tau_u > 0}} \phi(\tau_u, X_u(\tau_u)) \tag{5.18}$$

We will show that in the limit, this term goes to

$$\int_0^t \int_{\mathbb{R}} \phi(s,x)\Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu_s(dx)\right) \mu_s(dx)\, ds$$

which is the final term in Equation 5.17. We do this by introducing a martingale which will make this convergence clear.

Lemma 22. *Let $\phi \in C_0^\infty$. Define*

$$M_t^\eta = \int_0^t \int_{\mathbb{R}} \phi \, d\eta^N(x,s) - \int_0^t \frac{1}{N} \sum_{u \in A_s} \phi(s, X_u(s))\lambda(X_u(s)) \, ds \tag{5.19}$$

Then M_t^η is a martingale.

Proof. To show this, we fix a value s and define the random variable

$$f(t) = \mathbb{E}\left[M_t^\eta - M_s^\eta \middle| \mathcal{F}_s \right] \tag{5.20}$$

We will show that $f'(t) = 0$ for all $t > 0$. Since $f(s) = 0$, this will show that $f(t) = 0$ for all $t \geq s$.

We will use the definition of the derivative:

$$\begin{aligned}
f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \mathbb{E}\left[\int_t^{t+h} \int_{\mathbb{R}} \phi \, d\eta^N(x,s) - \int_t^{t+h} \frac{1}{N} \sum_{u \in A_s} \phi(s, X_u(s))\lambda(X_u(s)) \, ds \middle| \mathcal{F}_s \right] \\
&= \lim_{h \to 0} \frac{1}{Nh} \mathbb{E}\left[\mathbb{E}\left[\sum_{u \in A_{t+h} \setminus A_t} \phi(\tau_u, X_u(\tau_u)) - \int_t^{t+h} \sum_{u \in A_s} \phi(s, X_u(s))\lambda(X_u(s)) \, ds \middle| \mathcal{F}_t \right] \mathcal{F}_s \right]
\end{aligned}$$

We will work first on simplifying

$$\mathbb{E}\left[\sum_{u \in A_{t+h} \setminus A_t} \phi(\tau_u, X_u(\tau_u)) \middle| \mathcal{F}_t \right] \tag{5.21}$$

It will be more convenient to talk about these birth events in order, so we let τ^j be the jth birth event after time t. To avoid having to indicate which particle it was that branched at time τ^j, we will abuse notation slightly and write X^j to reference the particle that branched at time τ^j. Then Equation 5.21 expands to

$$
\begin{aligned}
&= \mathbb{E}[\phi(\tau^1, X^1(\tau^1))\mathbb{1}\{\tau^1 < t+h\}|\mathcal{F}_t] + \mathbb{E}\left[\sum_{j=2}^{\infty}\phi(\tau^j, X^j(\tau^j))\mathbb{1}\{\tau^j < t+h\}\middle|\mathcal{F}_t\right] \\
&= ① + ②
\end{aligned}
\tag{5.22}
$$

Looking at ②, we can see that it is of order h^2:

$$
\mathbb{E}\left[\sum_{j=2}^{\infty}\phi(\tau^j, X^j(\tau^j))\mathbb{1}\{\tau^j < t+h\}\middle|\mathcal{F}_t\right] \leq ||\phi||_\infty \mathbb{E}\left[\sum_{j=2}^{\infty}\mathbb{1}\{\tau^j < t+h\}\middle|\mathcal{F}_t\right]
$$

We know that the kth branch time of this process can be dominated by the kth branch time in a BBM with rate $M_\Lambda = ||\Lambda||_\infty$, so if $\tilde{\tau}^j$ is the jth branch time in a process of N_t BBMs with rate M_Λ, then we have

$$
\begin{aligned}
||\phi||_\infty \mathbb{E}\left[\sum_{j=2}^{\infty}\mathbb{1}\{\tau^j < t+h\}\middle|\mathcal{F}_t\right] &\leq ||\phi||_\infty \mathbb{E}\left[\sum_{j=2}^{\infty}\mathbb{1}\{\tilde{\tau}^j < h\}\middle|\mathcal{F}_t\right] \\
&= ||\phi||_\infty \mathbb{E}\left[\tilde{N}_h - N_t - \mathbb{1}\{\tilde{\tau}^1 < h\}\middle|\mathcal{F}_t\right] \\
&= ||\phi||_\infty \left(N_t \mathrm{e}^{M_\Lambda h} - N_t - \mathbb{P}(\geq 1 \text{ birth})\right) \\
&= ||\phi||_\infty \left(N_t M_\Lambda h + O(h^2) - (N_t M_\Lambda h + O(h^2))\right) \\
&= ||\phi||_\infty O(h^2)
\end{aligned}
$$

where $\tilde{N}_h$ is the number of particles alive at time h in the rate M_Λ process of N_t BBMs.

To determine ①, we will first determine

$$
\mathbb{E}\left[\phi(\tau^1, X^1(\tau^1))\middle|\mathcal{F}_t, \mathcal{W}_{t+h}\right] \tag{5.23}
$$

where $\mathcal{W}_{t+h}$ is the σ-algebra generated by the paths $\{X_u(s)\}_{u\in A_t}$ for $t \le s \le t+h$. This provides extra information, but does not entirely determine the expectation because the σ-algebra does not contain any information about the branch times of the particles. Conditioned on this information, the distribution of the first branch time becomes more straightforward.

$$\mathbb{E}\left[\phi(\tau^1, X^1(\tau^1))\middle|\mathcal{F}_t, \mathcal{W}_{t+h}\right] = \mathbb{E}\left[\sum_{u\in A_t}\int_t^{t+h}\phi(s, X_u(s))\mathbb{P}(\text{process branches 1st at } X_u(s))\,ds\middle|\mathcal{F}_t, \mathcal{W}_{t+h}\right]$$

$$= \mathbb{E}\left[\sum_{u\in A_t}\int_t^{t+h}\phi(s, X_u(s))\mathbb{P}(\text{1st birth by } X_u \text{ at } s)\mathbb{P}(\text{1st birth by } X_v \text{ after } s \text{ for all } v \neq u)\,ds\middle|\mathcal{F}_t, \mathcal{W}_{t+h}\right]$$

$$= \sum_{u\in A_t}\int_t^{t+h}\phi(s, X_u(s))\left(\lambda(X_u(s), \mu_s^N)\mathrm{e}^{-\int_t^s \lambda(X_u(r))\,dr}\right)\left(\mathrm{e}^{-\int_t^s \lambda(X_v(r))\,dr}\right)^{N_t-1}\,ds$$

Notice that each of the terms $\mathrm{e}^{-\int_t^s \lambda(X_u(r))\,dr} = 1+O(h)$ because $s-t<h$. Therefore, grouping the $O(h)$ terms inside the time integral, we get

$$= \sum_{u\in A_t}\int_t^{t+h}\phi(s, X_u(s))\lambda(X_u(s), \mu_s^N)(1+O(h))(1+O(h))^{N_t-1}\,ds$$

$$= \sum_{u\in A_t}\int_t^{t+h}\phi(s, X_u(s))\lambda(X_u(s), \mu_s^N) + O(h)\,ds$$

$$= \int_t^{t+h}\sum_{u\in A_t}\phi(s, X_u(s))\lambda(X_u(s))\,ds + O(h^2)$$

Notice that the constant in the $\mathrm{O}(h^2)$ depends on the paths X_k. This might be worrying once we are no longer conditioning on $\mathcal{W}_{t+h}$. But because λ is bounded and each $X_u(t)$ only appears in the constants in the form $\lambda(X_u(t))$, we know that this term is $O(h^2)$ regardless. So we have shown that 1 satisfies:

$$\mathbb{E}[\phi(\tau^1, X^1(\tau^1))\mathbb{1}\{\tau^1 < t+h\}|\mathcal{F}_t] = \mathbb{E}\left[\mathbb{E}\left[\phi(\tau^1, X^1(\tau^1))\mathbb{1}\{\tau^1 < t+h\}\middle|\mathcal{F}_t, \mathcal{W}_{t+h}\right]\middle|\mathcal{F}_t\right]$$
$$= \mathbb{E}\left[\int_t^{t+h} \sum_{u\in A_t} \phi(s, X_u(s))\lambda(X_u(s))\,ds + O(h^2)\middle|\mathcal{F}_t\right] \tag{5.24}$$

Because ② is $O(h^2)$, we can therefore say that

$$\mathbb{E}\left[\sum_{u\in A_{t+h}\setminus A_t} \phi(\tau_u, X_u(\tau_u))\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^{t+h} \sum_{u\in A_t} \phi(s, X_u(s))\lambda(X_u(s))\,ds\middle|\mathcal{F}_t\right] + O(h^2) \tag{5.25}$$

This now looks very similar to the term which is subtracted in Equation 5.19. Upon inspection, the only difference is that in Equation 5.19, the sum is taken over A_s, rather than A_t. That means that when a new particle is born, the term in M_t^η subtracts off the integral of its path. We have seen already that the contribution of multiple births is $O(h^2)$. Therefore, all that we need to consider is the contribution of the first birth. We now show that the integral of these contributions is $O(h^2)$ as well.

$$\mathbb{E}\left[\int_t^{t+h} \sum_{u\in A_s} \phi(s, X_u(s))\lambda(X_u(s))\,ds\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^{t+h} \sum_{u\in A_t} \phi(s, X_u(s))\lambda(X_u(s))\,ds\middle|\mathcal{F}_t\right]$$
$$+ \sum_{u\in A_t} \mathbb{E}\left[\int_{\tau^1}^{t+h} \phi(s, X^1(s))\lambda(X^1(s))\,ds\middle|\mathcal{F}_t\right] \tag{5.26}$$

Taking this integral $\int_{\tau^1}^{t+h} \phi(s, X^1(s))\lambda(X^1(s))\,ds$ and conditioning on the birth time

τ^1, we show that this integral is $O(h^2)$.

$$\begin{aligned}\mathbb{E}&\left[\int_{\tau^1}^{t+h} \phi(s, X^1(s))\lambda(X^1(s))\,ds\middle|\mathcal{F}_t\right]\\&\qquad\leq \mathbb{E}\left[\int_t^{t+h} M_\Lambda \mathrm{e}^{-M_\Lambda s}\int_s^{t+h} \phi(r, X^1(r))\lambda(X^1(r))\,dr\,ds\middle|\mathcal{F}_t\right]\end{aligned} \tag{5.27}$$

Bounding each term in the inner integral by its maximum value and evaluating gives that the expectation is $O(h^2)$.

Therefore,

$$\begin{aligned}f(t+h)-f(t) &= \frac{1}{N}\mathbb{E}\left[\int_t^{t+h} \sum_{u\in A_t} \phi(s, X_u(s))\lambda(X_u(s))\,ds\right.\\&\qquad\left.-\int_t^{t+h} \sum_{u\in A_t} \phi(s, X_u(s))\lambda(X_u(s))\,ds + O(h^2)\middle|\mathcal{F}_t\right]\\&= O(h^2)\end{aligned} \tag{5.28}$$

From this, we can see that $f'(t) = 0$ almost surely for all t. Therefore M_t^η is a martingale as desired. □

In addition to showing that this is a martingale, we need to show that it is well-behaved. We show in the next two lemmas that the second moment of this martingale goes to 0 as N goes to infinity. We first show that it has a finite second moment, and then use this to show that in fact the second moment goes to 0.

Lemma 23. *The second moment of M_t^η is finite for all $t \leq T$ and all N. That is*

$$\sup_{t\leq T} \mathbb{E}\left[(M_t^\eta)^2\right] < \infty \quad \textit{for all } N \tag{5.29}$$

Proof. To show this, we expand and bound the terms in $\mathbb{E}[(NM_t^\eta)^2]$.

$$\begin{aligned}
\mathbb{E}\left[(NM_t^\eta)^2\right] &= \mathbb{E}\left[\left(\sum_{u\in A_t\setminus A_0}\phi(\tau_u, X_u(\tau_u)) - \int_0^t \sum_{u\in A_s}\phi(s, X_u(s))\lambda(X_u(s))\,ds\right)^2\right]\\
&= \mathbb{E}\left[\left(\sum_{u\in A_t\setminus A_0}\phi(\tau_u, X_u(\tau_u))\right)^2\right]\\
&\quad - 2\mathbb{E}\left[\sum_{u\in A_t\setminus A_0}\phi(\tau_u, X_u(\tau_u))\int_0^t \sum_{u\in A_s}\phi(s, X_u(s))\lambda(X_u(s))\,ds\right]\\
&\quad + \mathbb{E}\left[\left(\int_0^t \sum_{u\in A_s}\phi(s, X_u(s))\lambda(X_u(s))\,ds\right)^2\right]\\
&= 1 + 2 + 3
\end{aligned} \tag{5.30}$$

We bound each term separately. For the first term,

$$\begin{aligned}
1 &\leq \mathbb{E}[(||\phi||_\infty N_t)^2]\\
&= ||\phi||_\infty \mathbb{E}[N_t^2]\\
&\leq ||\phi||_\infty^2 N^2 \mathrm{e}^{M_\Lambda t}(2\mathrm{e}^{M_\Lambda t} - 1)
\end{aligned} \tag{5.31}$$

We bound 2 in the following way.

$$\begin{aligned}
|2| &\leq 2\mathbb{E}\left[||\phi||_\infty N_t \int_0^t \sum_{k\in A_s}|\phi(s, X_k(s))\lambda(X_k(s))|\,ds\right]\\
&\leq 2\mathbb{E}\left[||\phi||_\infty N_t \cdot M_\Lambda ||\phi||_\infty N_t t\right]\\
&\leq 2||\phi||_\infty^2 M_\Lambda t \mathbb{E}[N_t^2]\\
&\leq 2||\phi||_\infty^2 M_\Lambda t N^2 \mathrm{e}^{M_\Lambda t}(2\mathrm{e}^{M_\Lambda t} - 1)
\end{aligned} \tag{5.32}$$

For (3), we have

$$\begin{aligned}
(3) &\leq \mathbb{E}\left[(M_\Lambda t||\phi||_\infty N_t)^2\right] \\
&\leq M_\Lambda^2||\phi||_\infty^2 t^2\mathbb{E}[N_t^2] \\
&\leq M_\Lambda^2||\phi||_\infty^2 t^2 N^2 \mathrm{e}^{M_\Lambda t}(2\mathrm{e}^{M_\Lambda t}-1)
\end{aligned} \tag{5.33}$$

It is clear that each part of the equation is bounded by $C_i N^2$ where C_i depends on time but not on N. Therefore, $\mathbb{E}[(M_t^\eta)^2] \leq C(t)$, a constant which does not depend on N. So $\sup_{t\leq T}\mathbb{E}\left[(M_t^\eta)^2\right] \leq C(T)$ as desired. □

Lemma 24. $\mathbb{E}[(M_t^\eta)^2] \to 0$ *as* $N \to \infty$*. Therefore,*

$$\left|\int_0^t\int_{\mathbb{R}}\phi\, d\eta^N(x,s) - \int_0^t \sum_{u\in A_s}\phi(s,X_u(s))\lambda(X_u(s))\,ds\right| \to 0 \qquad \textit{in } L^2$$

Proof. To help with notation, call $Z_t = \frac{1}{N}\sum_{k\in A_t\setminus A_0}\phi(\tau_k, X_k(\tau_k))$.

From Lemmas 22 and 23, we know that M_t^η is martingale with locally finite second moment, and therefore its quadratic variation exists locally and

$$(M_t^\eta)^2 - [M^\eta, M^\eta]_t \tag{5.34}$$

is a local martingale. Noticing that $\int_0^t \sum_{k\in A_s}\phi(s,X_k(s))\lambda(X_k(s))\,ds$ is continuous and has finite variation, we can see that

$$[M^\eta, M^\eta]_t = [Z,Z]_t \tag{5.35}$$

Because Z_t is a jump process, its quadratic variation is the sum of the jump sizes. So we have that

$$[Z,Z]_t = \sum_{u\in A_t\setminus A_0}\frac{1}{N^2}\phi^2(\tau_u, X_u(\tau_u)) \tag{5.36}$$

Noticing that 5.36 is similar to Z_t with $\frac{1}{N}\phi^2$ in place of ϕ, we can use the same argument as in Lemma 22 to say that

$$
\begin{aligned}
\overline{Z}_t &= \sum_{u \in A_t \setminus A_0} \frac{1}{N^2}\phi^2(\tau_u, X_u(\tau_u)) - \int_0^t \frac{1}{N} \sum_{u \in A_s} \frac{1}{N}\phi^2(s, X_u(s)\lambda(X_u(s))\, ds \\
&= [Z, Z]_t - \frac{1}{N^2}\int_0^t \sum_{u \in A_s} \phi^2(s, X_u(s))\lambda(X_u(s))\, ds \\
&= [M, M]_t - \frac{1}{N^2}\int_0^t \sum_{u \in A_s} \phi^2(s, X_u(s))\lambda(X_u(s))\, ds
\end{aligned}
\tag{5.37}
$$

is a martingale. In particular, this means that

$$
\begin{aligned}
\mathbb{E}\left[(M_t^\eta)^2\right] &= \mathbb{E}[M, M]_t \\
&= \mathbb{E}\left[\frac{1}{N^2}\int_0^t \sum_{u \in A_s} \phi^2(s, X_u(s))\lambda(X_u(s))\, ds\right] \\
&\leq \frac{1}{N^2}\mathbb{E}\left[\int_0^t N_t M_\phi^2 M_\Lambda\, ds\right] \\
&\leq \frac{\mathrm{e}^{M_\Lambda t} M_\phi^2 M_\Lambda t}{N}
\end{aligned}
\tag{5.38}
$$

where we have once again defined $M_\Lambda = ||\Lambda||_\infty, M_\phi = ||\phi||_\infty$. Therefore, we have our desired result that for any fixed t,

$$\mathbb{E}\left[(M_t^\eta)^2\right] \to 0 \quad \text{in } L^2 \text{ as } N \to \infty$$

And as such,

$$\left| \int_0^t \int_{\mathbb{R}} \phi(s, x)\, d\eta^N(s, x) - \int_0^t \int_{\mathbb{R}} \phi(s, x)\lambda(x)\, d\mu_s^N\, ds \right| \to 0 \quad \text{in } L^2$$

$\square$

5.4.5 A Limit of the Weak Solution Equation

We have now shown that $\{\mu_t^N\}$ is tight, that every μ_t^N must satisfy Equation 5.16, and have proven almost all the lemmas necessary to take a limit of Equation 5.16. Now we will show that every convergent subsequence converges to the same limit, one satisfying Equation 5.17; this would imply that the entire sequence converges to that limit. Therefore, existence and uniqueness of the limit will be reduced to a question of uniqueness of weak solutions to the limiting PDE.

We want to show convergence of integrals of the form $\int_{\mathbb{R}} g(x,s)\,d\nu_n(s,x)$ and $\int_0^t \int_{\mathbb{R}} g(x,s)\,d\nu_n(s,x)\,ds$ for $g(x,s)$ a test function in C_0^∞. To do this, we need to verify that all the integrals are continuous functions of the measure-valued processes.

As indicated in [7, Chapter 3], weak convergence of the probability measure ν_n on the space $D([0,T],\mathcal{M}_+)$ does not necessarily imply weak convergence of ν_n^t, the projection of the measures at time t for some $t \in [0,T]$. In particular, for weak convergence of the finite-dimensional distributions, we require an additional condition of continuity, P-a.s., at the selected times. To this end, define $\pi_t : D([0,T],\mathcal{M}_+) \to \mathcal{M}_+$ as the projection of the cadlag process at time t: $\pi_t(\nu) = \nu(t)$ and let T_P be the set of $t \in [0,T]$ such that π_t is continuous except at a set of P-measure 0. It is known that $\{0,T\} \subset T_P$ always. If $0 < t < T$, then π_t is continuous at $\nu \in D([0,T],\mathcal{M}_+)$ if and only if ν is continuous at t a.s. Therefore, $t \in T_P$ if and only if $P(J_t) = 0$, that is, if the probability that the process jumps at t is 0. Because there are no distinguished times in this process and the probability of a jump at any fixed time is 0 for each N, Lemma 25 follows:

Lemma 25. *$T_P = [0,T]$ for the weakly-dependent BBM process. Therefore, if $\mu_t^N \Rightarrow \mu_t$ as a sequence of measure-valued processes, then $\mu_s^N \Rightarrow \mu_s$ as a sequence of measures for all $s \in [0,T]$.*

Proof. The fact that $T_P = [0,T]$ follows from the fact that the ν_n probability of a jump at time t is 0 for all $t \in [0, T]$, and because there can be at most countably many of these jumps, no jumps can develop in the limit. The statement about the weak convergence of ν_n^t therefore follows from [7, Chapter 3.13]. $\square$

Lemma 26. *Let* $f : D([0, T], \mathcal{M}_+) \to D([0, T], \mathbb{R})$, *with* $f(S)(t) = \int_{\mathbb{R}} g(x, t) S(t, dx)$. *If* $g(\cdot, s) \in C_c^\infty(\mathbb{R})$ *for any* s, *then* f *is a continuous function.*

Proof. Let d_M be the Skorohod distance on $D([0, T], \mathcal{M}_+)$ and $d_{\mathbb{R}}$ be the Skorohod distance on $D([0, T], \mathbb{R})$. Fix $\varepsilon > 0, Y \in D([0, T], \mathcal{M}_+)$. Notice that because g is smooth and compactly supported, it is Lipschitz and uniformly continuous. Therefore, we can define $|g|_L$ to be the Lipschitz constant for g and δ_1 to be a constant such that if $|t - s| < \delta_1$, then $|g(t, x) - g(s, x)| < \frac{\varepsilon}{2 \sup_{t \leq T} \int_{\mathbb{R}} Y(t, dx)}$.

Let $\delta = \min\left(\frac{\varepsilon}{2|g|_L}, \delta_1, \varepsilon\right)$. For any X such that $d_M(S, Y) \leq \delta$, we have the following. By the definition of d_M, there exists an continuous, increasing, bijective function $\lambda : [0, T] \to [0, T]$ such that

$$\begin{aligned} &\sup_{t \leq T} |t - \lambda t| \leq \delta \\ &\sup_{t \leq T} ||S(t) - Y(\lambda t)||_{Wass} \leq \delta \end{aligned} \tag{5.39}$$

Using that same λ, we see that

$$\begin{aligned} |f(S)(t) - f(Y)(\lambda t)| &= \left| \int_{\mathbb{R}} g(t, x) S(t, dx) - \int_{\mathbb{R}} g(\lambda t, x) Y(\lambda t, dx) \right| \\ &\leq \left| \int_{\mathbb{R}} g(t, x) S(t, dx) - \int_{\mathbb{R}} g(t, x) Y(\lambda t, dx) \right| + \\ &\quad \left| \int_{\mathbb{R}} g(t, x) Y(\lambda t, dx) - \int_{\mathbb{R}} g(\lambda t, x) Y(\lambda t, dx) \right| \end{aligned} \tag{5.40}$$

Because $\frac{g(t,x)}{|g|_L}$ is a Lipschitz function with Lipschitz constant 1, we know that by the definition of the Wasserstein distance, we have

$$\left| \int_{\mathbb{R}} \frac{g(t,x)}{|g|_L} (S(t,dx) - Y(\lambda t, dx)) \right| \leq d_M(S(t), Y(\lambda t)) \leq \delta \tag{5.41}$$

Therefore, (5.40) simplifies to

$$|f(S)(t) - f(Y)(\lambda t)| \leq |g|_L \delta + \left| \int_{\mathbb{R}} (g(t,x) - g(\lambda t, x)) Y(\lambda t, dx) \right| \tag{5.42}$$

Because $|t - \lambda t| \leq \delta \leq \delta_1$, we know that $|g(t,x) - g(\lambda t, x)| \leq \frac{\varepsilon}{2 \sup_{t \leq T} \int_{\mathbb{R}} Y(t,dx)}$. Therefore, we get that the last term in (5.42) can be bounded by

$$\begin{aligned} |f(S)(t) - f(Y)(\lambda t)| &\leq |g|_L \delta + \frac{\varepsilon}{2 \sup_{t \leq T} \int_{\mathbb{R}} Y(t,dx)} \int_{\mathbb{R}} Y(\lambda t, dx) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{5.43}$$

Notice that this bound is independent of t. Therefore,

$$\sup_{t \leq T} |f(S)(t) - f(Y)(\lambda t)| \leq \varepsilon \tag{5.44}$$

Because we used the same λ, we know that

$$\sup_{t \leq T} |t - \lambda t| \leq \delta \leq \varepsilon \tag{5.45}$$

So by the definition of $d_{\mathbb{R}}$, we can see that $d_{\mathbb{R}}(f(S), f(Y)) \leq \varepsilon$. Therefore f is continuous at Y for each $Y \in D([0,T], \mathcal{M}_+)$. $\square$

Lemma 27. *Let a be an element in $D([0,T],\mathbb{R})$. Then $h(a)(t) = \int_0^t a(s)\, ds$ is a continuous function of a for all t. And therefore, $h(a)$ is a continuous map $h : D([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$.*

Proof. Fix an $\varepsilon > 0$. Fix a point $a \in D([0,T],\mathbb{R})$. We want to show that there exists a δ such that if $b \in D([0,T],\mathbb{R})$ is chosen such that $d_{\mathbb{R}}(a,b) < \delta$, then $\left| \int_0^t a(s)\, ds - \int_0^t b(s)\, ds \right| < \varepsilon$. Before choosing a δ, we define the following:

Let $a^\delta(s) = \sup_{|s-s'|\le\delta} a(s')$ and $a_\delta(s) = \inf_{|s-s'|\le\delta} a(s')$. Notice that $a^\delta(s), a_\delta(s)$ converge to $a(s)$ as $\delta \to 0$ wherever a is a continuous (and therefore almost everywhere). Also notice that $a_\delta(s) \le a(s) \le a^\delta(s)$, and that the convergence of $a^\delta(s)$, $a_\delta(s)$ is monotonic. Therefore, by the monotonic convergence theorem, we know that

$$\int_0^t a^\delta(s)\,ds \to \int_0^t a(s)\,ds$$

$$\int_0^t a_\delta(s)\,ds \to \int_0^t a(s)\,ds$$

for all t. Therefore, we can choose a δ_1 such that $\left|\int_0^t a^{\delta_1}(s)\,ds - \int_0^t a_{\delta_1}(s)\,ds\right| \le \frac{\varepsilon}{2}$. Pick $\delta = \min(\delta_1, \frac{\varepsilon}{2T})$ and let $b \in D([0,T],\mathbb{R})$ such that $d_{\mathbb{R}}(a,b) \le \delta$. Then there exists a λ such that $|a(\lambda(s)) - b(s)| \le \frac{\varepsilon}{2T}$. Then we have the following:

$$\begin{aligned}
\left|\int_0^t a(s) - b(s)\,ds\right| &= \left|\int_0^t a(s) - a(\lambda(s)) + a(\lambda(s)) - b(s)\,ds\right| \\
&\le \left|\int_0^t a(s)\,ds - \int_0^t a(\lambda(s))\,ds\right| + \int_0^t |a(\lambda(s)) - b(s)|\,ds \\
&\le \left|\int_0^t a(s)\,ds - \int_0^t a(\lambda(s))\,ds\right| + \frac{\varepsilon}{2} \\
&\le \left|\int_0^t a^{\delta_1}(s)\,ds - \int_0^t a_{\delta_1}(s)\,ds\right| + \frac{\varepsilon}{2}
\end{aligned}$$

where the last inequality comes from the fact that $|\lambda(s) - s| \le \delta \le \delta_1$, so we know that $a_{\delta_1}(s) \le a(\lambda(s)) \le a^{\delta_1}(s)$. Therefore,

$$\left|\int_0^t a(s) - b(s)\,ds\right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, we have that $h(a)(t)$ is continuous for each t.

All that remains to show that $h(a)$ is a continuous function from $D([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$. Suppose $\{a_k\}$ is a convergent sequence in $D([0,T],\mathbb{R})$. Then we have shown that $\{\int_0^t a_k(s)\,ds\}$ is a convergent sequence for every t and if $a_k \to a$, then $\int_0^t a_k(s)\,ds \to \int_0^t a(s)\,ds$. But $\{\int_0^t a_k(s)\,ds\}$ is a sequence of continuous functions on a compact set which converge pointwise to a continuous function. Therefore, the convergence is uniform. $\square$

We use the continuity of f to show convergence of the desired integrals.

Lemma 28. *If P_n, P are probability measures on $D([0,T],\mathcal{M}_+)$ such that $P_n \Rightarrow P$, $\mathbb{P}_n(t) \Rightarrow P(t)$ for all t, and S_n, S are the random variables of $D([0,T],\mathcal{M}_+)$ with law P_n, P respectively, then $\int_{\mathbb{R}} g(x,t)S_n(t,dx) \Rightarrow \int_{\mathbb{R}} \phi(t,x)S(t,dx)$ and $\int_0^t \int_{\mathbb{R}} g(x,s)S_n(s,dx)\,ds \Rightarrow \int_0^t \int_{\mathbb{R}} g(s,x)S(s,dx)\,ds$ weakly.*

Proof. Because f is continuous and $S_n(t)$ converges in distribution to $S(t)$ for all t, then $f(S_n(t)) \Rightarrow f(S(t))$ weakly by the continuous mapping theorem. To see the convergence for the time integrals, we note that $\int_0^t \int_{\mathbb{R}} g(x,s)\,S(s,dx)\,ds$ is a composition of continuous functions and is therefore also continuous. So again by the continuous mapping theorem, we have that if $S_n \Rightarrow S$ weakly, then
$\int_0^t \int_{\mathbb{R}} g(x,s)\,S_n(s,dx)\,ds \Rightarrow \int_0^t \int_{\mathbb{R}} g(x,s)\,S(s,dx)\,ds$ weakly. $\square$

We have essentially proven our desired hydrodynamic limit. All that remains is to put the last pieces together.

Proof of Theorem 16. Using Lemmas 21, 24, and 28, we take the limit of Equation 5.16 along any convergent subsequence $\{\mu_t^k\}$ and get convergence in distribution.

$$\lim_{k\to\infty} \langle \phi, \mu_s^k \rangle|_0^t = \lim_{k\to\infty} \int_0^t \int_{\mathbb{R}} \left(\phi_t + \frac{1}{2}\phi_{xx}\right) d\mu_s^k\,ds + \int_0^t \int_{\mathbb{R}} \phi\, d\eta^N(x,s) + \frac{M_t}{k}$$

$$\lim_{k\to\infty} \langle \phi, \mu_s^k \rangle|_0^t = \lim_{k\to\infty} \int_0^t \int_{\mathbb{R}} \left(\phi_t + \frac{1}{2}\phi_{xx}\right) d\mu_s^k\,ds + \int_0^t \int_{\mathbb{R}} \phi\lambda(x)\, d\mu_s^k\,ds + \frac{M_t}{k} \quad (5.46)$$

$$\langle \phi, \mu_s \rangle|_0^t = \int_0^t \int_{\mathbb{R}} \left(\phi_t + \frac{1}{2}\phi_{xx} + \phi\lambda(x)\right) d\mu_s\,ds$$

Recall the definition of $\lambda(x)$ is such that this compact notation actually represents

$$\int_0^t \int_{\mathbb{R}} \phi\lambda \, d\mu_s \, ds = \int_0^t \int_{\mathbb{R}} \phi(s,x)\Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu_s(dx)\right) \mu_s(dx)\, ds$$

This expanded representation makes it clear that the limit along any convergent subsequence, $\mu_t(dx)$, is the weak solution to the PDE (5.5). Under the hypothesis that there is a unique solution, we can conclude that the whole sequence $\{\mu_t^N\}$ converges and has a limit μ_t. Because the solution to the equation is smooth, we can say that $\mu_t(dx) = u(x,t)\, dx$, with $u(x,t)$ solving Equation 5.5 as desired. $\square$

6

Conclusion

In this dissertation, we have analyzed the asymptotic behavior of three branching interacting particle systems. We summarize the results and some open questions associated with each process.

First, we studied the N-BBM process with selection according to the fitness function $x + \Psi(x)$, where Ψ is periodic. We proved the existence of a long-time limiting speed of the system and the existence of a stationary distribution in a moving frame. Further work is ongoing to study the positivity and value of the speed. Additionally, the speed was proven in the case where the initial distribution of particles was chosen according to the invariant distribution; it remains open to show that if the process began in a different distribution that the speed would still exist. This would require further study of convergence of the process to the invariant measure in the moving frame.

In the next chapter, we looked at the N-BBM process with a symmetrically decaying fitness function which had a single local maximum at the origin. Study of this process was inspired by the behavior of the first process while stuck in a local fitness peak. We showed a hydrodynamic limit of the process, where convergence was obtained in the Kolmogorov distance almost surely and in L^1. This limiting measure was shown to satisfy a free boundary PDE. Several further directions of inquiry are open. For instance, the particles are required to begin in a symmetric configuration. This is necessitated by a technical constraint, but is not likely to be necessary for a similar result to hold. A second open line of inquiry would be to consider a fitness function which decayed monotonically away from the origin but which was not not symmetric. This condition was necessary here as a part of the proof technique which treated particles on either side of the origin were indistinguishable. It is an open question to show that the result holds under these more general conditions.

Finally, we studied a branching process in which the branch rate of a particle was a function of the empirical measure of the process. We showed convergence weakly to a limit which satisfies a non-local PDE, when a unique solution exists. Further inquiry can be done into precisely the conditions which ensure that such a solution exists. Additionally, the rate was required to be bounded and positive; further study to loosen these conditions would allow for interesting cases to be considered, including the motivating process where the branch rate is a function of a particle's distance from the empirical mean. However, as indicated, there are many difficulties in making this generalization. In particular, controlling particle growth and ensuring there is no finite time explosion becomes a difficult problem that needs additional study.

Appendix A

Assorted Facts and Proofs

A.1 Hydrodynamic Limit of a BBM

The purpose of this section is to prove the following hydrodynamic limit.

Theorem 29. *Let $X^N(t)$ be a particle system beginning with N binary, rate 1 branching Brownian motions in $\mathbb{R}$ where the initial positions of each particle chosen independently and distributed according to the probability density $\rho(x)$. Let $\mu_t^N(x) = \dfrac{1}{N}\sum_{k=1}^{N_t} \delta_{X_k(t)}$, where N_t is the number of particles alive at time t. Then $\lim_{N\to\infty} \mu_t^N(dx) = u(x,t)\,dx$ weakly, where $u(x,t)$ is the solution to the PDE*

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + u \quad\quad x \in \mathbb{R}, t > 0 \\ u(x,0) &= \rho(x) \end{aligned} \tag{A.1}$$

The proof techniques used here are versatile and form the basic structure of many proofs of this type. The theorems below are not original; they can be found in [22], [16], [7] for instance. We begin by recalling several theorems which will be used and then prove the theorem in small pieces.

We call a collection of probability measures Π *tight* if for each $\varepsilon > 0$, there exists a compact set K such that $P(K^c) \leq \varepsilon$ for each $P \in \Pi$.

We call a collection of probability measures Π *relatively compact* if each sequence $\{P_n\}$ of measures in Π has a further subsequence $\{P_{n_i}\}$ which converges weakly.

Theorem 30 ([7] Prokhorov's Theorem)**.** *Let S be a complete and separable space. Then for any collection of probability measures Π, Π is tight if and only if it is relatively compact.*

We also have a theorem which says that if a sequence of probability measures is tight and each subsequence which converges has the same limit, then we know that the entire sequence converges to that limit.

Theorem 31. *If $\{P_n\}$ is tight and each subsequence which converges weakly at all converges weakly to the measure P, then $P_n \Rightarrow P$.*

These theorems together outline a clear path for determining weak convergence of a sequence of probability measures. First, one must show that the sequence of empirical measures is tight (in the appropriate space). By Prokhorov's Theorem, this will then guarantee that the sequence is relatively compact. Then, one looks to find a characterization of the limit object of a subsequence which will prove that each such limit object must agree. This is often done by relating the limit objects to PDE solutions or to a particular martingale. This way, uniqueness can be obtained through other analysis methods and combined with Theorem 31 to give the desired weak convergence.

When we are dealing with convergence in these cases, we choose to view our objects as measure-valued processes; that is, our process is an object in $D([0,T], \mathcal{M}_+)$, where $\mathcal{M}_+$ is the space of positive, finite measures. Luckily, this is a complete, separable metric space.

We endeavor to prove tightness in this space. A sufficient condition for tightness in $D([0,T],\mathcal{M}_+)$ was given by Roelly-Coppoletta in [22]. It relates tightness in the Skorohod space of measure-valued processes to tightness in the Skorohod space of real-valued processes.

Theorem 32. *Let $\{P_n\}_n$ be a sequence of probability measures on $D([0,T],\mathcal{M}_+)$ and let $\{f_k\}_k$ be a dense sequence of functions in $C_0(\mathbb{R},\mathbb{R})$. Define $\pi_{f_k}P_n$ to be the pushforward measure through f_k*

$$\pi_{f_k}P_n = \int_{\mathbb{R}} f_k\, P_n \tag{A.2}$$

If, for each k in $\mathbb{N}$, $\{\pi_{f_k}P_n\}_n$ is a tight sequence of probabilities on the space $D([0,T],\mathbb{R})$, then $\{P_n\}_n$ is tight on $D([0,T],\mathcal{M}_+)$.

From there, we can refer to the many theorems that give sufficient conditions for tightness in the Skorohod space. One such theorem which will be useful for our purposes is the characterization via Aldous (see for instance [16] for the version below or [1] for the original presentation).

Theorem 33 (Aldous Condition)**.** *For each ε,η,m, there exists a δ_0 and n_0 such that if $\delta\le\delta_0$ and $n\ge n_0$ and if τ is a discrete X^n-stopping time satisfying $\tau\le m$, then*

$$P(|X^n_{\tau+\delta}-X^n_\tau|\ge\eta)\le\varepsilon \tag{A.3}$$

Theorem 34. *If the Aldous condition holds and for each t, the laws of $\{\Delta X^n_t\}_n$, where $\Delta X^n_t = X^n_t - \lim\limits_{s\to t^-} X^n_s$, form a tight sequence, then the distributions of the sequence $\{X^n\}_n$ are tight on $D([0,T],\mathbb{R})$.*

So let us take a function $f\in C_0$ and define $M_f = ||f||_\infty$. Then we consider the sequence of probability measures $\{\pi_f\mu^n_t\}_n$. The measure $\pi_f\mu^n_t$ is the law of the process $Y^n_t = \frac{1}{n}\sum_{k=1}^{N_t} f(X_k(t))$.

Lemma 35. *The Y_t processes satisfy the Aldous condition.*

Proof. We let A_t be the collection of particles alive at time t and $N_t = |A_t|$ be the number of particles alive at time t. Fix an ε, η, m and τ a Y^n-stopping time with $\tau \leq m$. We look to bound

$$P(|Y^n_{\tau+\delta} - Y^n_\tau| \geq \eta) = P\left(\left|\frac{1}{n}\sum_{u\in A_{\tau+\delta}} f(X_u(\tau+\delta)) - \frac{1}{n}\sum_{u\in A_\tau} f(X_u(\tau))\right| \geq \eta\right)$$
$$\leq P\left(\frac{1}{n}\sum_{u\in A_\tau} |f(X_u(\tau+\delta)) - f(X_u(\tau))| + \frac{1}{n}\sum_{u\in A_{\tau+\delta}\setminus A_\tau} |f(X_u(\tau+\delta)| \geq \eta\right) \tag{A.4}$$

What this says is that we need to control the amount that the particles move in the interval $[\tau, \tau+\delta]$ and we need to control the number of new particles which are born in that interval.

For ease of reference, let E be the event that $|Y^n_{\tau+\delta} - Y^n_\tau| \geq \eta$. We first explain how to choose δ, n_0.

We know that the number of particles alive at time τ is a random variable whose distribution is the same as the distribution of the sum of n independent $\mathrm{Geo}(\mathrm{e}^{-\tau})$ random variables. Therefore, $\mathrm{Var}(N_\tau) = n\mathrm{e}^{2\tau}(1-\mathrm{e}^{-\tau})$ and we can fix an a such that

$$\begin{aligned}\mathbb{P}(|N_\tau - n\mathrm{e}^\tau| \geq na) &\leq \frac{\mathrm{Var}(N\tau)}{a^2n^2} \\ &= \frac{\mathrm{e}^{2\tau}(1-\mathrm{e}^{-\tau})}{a^2n} \\ &\leq \frac{\varepsilon}{3}\end{aligned} \tag{A.5}$$

where the first inequality is Chebyshev's inequality. Define $M_N = n\mathrm{e}^\tau + na$.

Let A be the event $N_\tau < M_N$. We have selected a to ensure that $P(A^c) \leq \frac{\varepsilon}{3}$.

Because $f \in C_0$, for each $u \in A_\tau$, there exists a Δx_u such that if $d(X_u(\tau), y) \leq \Delta x_u$, then $d(f(X_u(\tau)), f(y)) \leq \frac{\eta}{2M_N}$. Let $\Delta x = \min_{u\in A_\tau} \Delta x_u$. Pick δ_1 small enough such that $n\mathbb{P}(|B(\delta)-B(0)| \geq \Delta x) \leq \frac{\varepsilon}{3}$. Let B be the event that the N_τ Brownian particles which are alive at time τ all move less than Δx.

Choose δ_2 small enough so that each particle alive at time τ has at most one offspring during the interval $[\tau, \tau+\delta_2]$. Precisely, choose δ_2 small enough so that $\mathbb{P}(|F_u^\tau(\tau+\delta_2)| > 2 \text{ for some } u \in A_\tau) \leq \frac{\varepsilon}{3}$, where $F_u^\tau(\tau+\delta_2) = \{v \in A_{\tau+\delta_2} \bigm| u < v\}$. Let C be the event that all particles alive at time τ have at most one offspring by time $\tau+\delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. Finally, pick $n_0 > \frac{2M_n M_f}{\eta}$.

$$\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}(E|A)\mathbb{P}(A) + \mathbb{P}(E|A^c)\mathbb{P}(A^c) \\
&\leq \mathbb{P}(E|A) + \mathbb{P}(A^c) \\
\mathbb{P}(E|A) &= \mathbb{P}(E|A\cap B)\mathbb{P}(B|A) + \mathbb{P}(E|A\cap B^c)\mathbb{P}(B^c|A) \\
&\leq \mathbb{P}(E|A\cap B) + \mathbb{P}(B^c|A) \\
\mathbb{P}(E|A\cap B) &= \mathbb{P}(E|A\cap B\cap C)\mathbb{P}(C|A\cap B) + \mathbb{P}(E|A\cap B\cap C^c)\mathbb{P}(C^c|A\cap B) \\
&\leq \mathbb{P}(E|A\cap B\cap C) + \mathbb{P}(C^c|A\cap B)
\end{aligned} \tag{A.6}$$

Therefore, we have that

$$\begin{aligned}
\mathbb{P}(E) &\leq \mathbb{P}(E|A\cap B\cap C) + \mathbb{P}(A^c) + \mathbb{P}(B^c|A) + \mathbb{P}(C^c|A\cap B) \\
&\leq \mathbb{P}(E|A\cap B\cap C) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\end{aligned} \tag{A.7}$$

All that remains is to show that $\mathbb{P}(E|A\cap B\cap C) = 0$. Consider the relevant sum:

$$\frac{1}{n}\sum_{u\in A_\tau} |f(X_u(\tau+\delta)) - f(X_u(\tau))| + \frac{1}{n}\sum_{u\in A_{\tau+\delta}\setminus A_\tau} |f(X_u(\tau+\delta))| \tag{A.8}$$

Given A, B, C means we know that $N_\tau < ne^\tau + na$, each particle alive at time τ moves less than Δx, where Δx is chosen so that the change in f can be at most $\frac{\eta}{2M_N}$, and each particle has at most one offspring in $[\tau, \tau+\delta]$. So for $n \geq n_0 > \frac{2M_N M_f}{\eta}$, we have that

$$
\begin{aligned}
\frac{1}{n}\sum_{u\in A_\tau}|f(X_u(\tau+\delta))-f(X_u(\tau))|+\frac{1}{n}\sum_{u\in A_{\tau+\delta}\setminus A_\tau}|f(X_u(\tau+\delta)|& \\
&\le \frac{1}{n}M_N\frac{\eta}{2M_N}+\frac{1}{n}M_NM_f \\
&\le \frac{\eta}{2}+\frac{M_nM_f}{n_0} \\
&\le \frac{\eta}{2}+\frac{\eta}{2} \\
&=\eta
\end{aligned}
\tag{A.9}
$$

Therefore, $\mathbb{P}(E|A\cap B\cap C)=0$. Plugging this back in to A.7, we get that

$$
\mathbb{P}(|Y^n(\tau+\delta)-Y^n(\tau)|\ge\eta)\le\varepsilon \tag{A.10}
$$

for $n\ge n_0$ as desired. Therefore, the collection of processes $\{Y^n\}$ satisfies the Aldous condition. □

Lemma 36. *$\{Y^n\}$ are tight in $D([0,T],\mathbb{R})$.*

Proof. As we have shown that the processes satisfy the Aldous condition, all that remains to apply Theorem 34 is to show that the laws of the jump process at any time t is tight. But notice that $\Delta Y^n_t\le M_f$ for all times t, because a jump in Y^n represents the addition of a new particle. As only one particle is added at a time with probability 1, and the jump is of size $f(X_k(t))$, where X_k is the branching particle, we know that the jump distribution is in fact bounded. Therefore, it is tight and we can apply Theorem 34 to say that the sequence of processes $\{Y^n\}$ are tight. □

This lemma, combined with Theorem 32, show that the laws of the processes $\{X^n\}$ are tight on $D([0,T],\mathcal{M}_+)$.

Finally, we now show that any limit of the process must be a weak solution of the PDE A.1. Let $\phi(x)\in C_c^\infty(\mathbb{R},\mathbb{R})$ be a test function. Then consider a subsequence

$\{\mu_t^{s(j)}(dx)\}$. Because the sequence $\{\mu_t^N(dx)\}$ is tight, we know that this subsequence has a limit, call it $\mu_t(dx)$.

$$\int_{\mathbb{R}} \phi(x)\, \mu_t^{N_j}(dx) = \frac{1}{N_j} \sum_{k=1}^{N_j} \sum_{i=1}^{N^k(t)} \phi(X_i^k(t))$$

Taking the limit of both sides, we see that

$$\lim_{j\to\infty} \int_{\mathbb{R}} \phi(x)\, \mu_t^{N_j}(dx) = \lim_{j\to\infty} \frac{1}{N_j} \sum_{k=1}^{N_j} \sum_{i=1}^{N^k(t)} \phi(X_i^k(t))$$

$$\int_{\mathbb{R}} \phi(x)\, \mu_t(dx) = \mathbb{E}\left[\sum_{i=1}^{N_j^1(t)} \phi(X_i^1(t))\right]$$

by the law of large numbers. Applying the many-to-one lemma, we see that

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{N_j^1(t)} \phi(X_i^1(t))\right] &= \mathrm{e}^t \mathbb{E}_\rho[\phi(B(t))] \\ &= \mathrm{e}^t \int_{\mathbb{R}} \int_R \phi(y)\rho(x)\Phi(x-y,t)\, dx\, dy \end{aligned}$$

where $\Phi(x,t) = \frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-x^2/2t}$ is the heat kernel.

Therefore, using the fact that $\rho \star \Phi$ solves the heat equation and the fact that if $w(x,t)$ is a solution to the heat equation, then $\mathrm{e}^t w(x,t)$ solves the heat equation with growth, we can see that this limit is a weak solution to the equation

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + u \quad x \in \mathbb{R}, t > 0 \\ u(x,0) &= \rho(x) \end{aligned} \tag{A.11}$$

Therefore, we know that the limit of the subsequence must be a weak solution to the heat equation. Because of the smoothing property of the heat equation, we can in fact say that the resulting limit has a density which is a strong solution to the heat equation. Because the solution to the heat equation is unique under sufficient growth conditions, we can apply Theorem 31 to say that the measures converge weakly to $u(x,t)\,dx$ where $u(x,t)$ solves the heat equation, as desired.

A.2 Distribution of the Size of a BBM

Theorem 37. *Let N_t be the size of a rate λ binary branching Brownian motion at time t. Then*

$$N_t \overset{d}{=} \mathrm{Geo}(\mathrm{e}^{-\lambda t})$$

That is, $\mathbb{P}(N_t = k) = (1-\mathrm{e}^{-\lambda t})^{k-1}(\mathrm{e}^{-\lambda t})$.

Proof. We know that the characteristic function of a geometric random variable with parameter p is

$$f(p,\theta) = \mathbb{E}[\mathrm{e}^{i\theta X}] = \frac{p\mathrm{e}^{i\theta}}{1-(1-p)\mathrm{e}^{i\theta}} \tag{A.12}$$

We first find an equality satisfied by $\mathbb{E}[\mathrm{e}^{i\theta N_t}]$ by conditioning on the first branch time τ.

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{i\theta N_t}] &= \mathbb{E}[\mathrm{e}^{i\theta N_t}|\tau > t]\mathbb{P}(\tau > t) + \mathbb{E}[\mathrm{e}^{i\theta N_t}|\tau \le t]\mathbb{P}(\tau \le t) \\
&= \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \int_0^t \mathbb{E}[\mathrm{e}^{i\theta N_t}|\tau = s]\mathbb{P}(\tau = s)\,ds \\
&= \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \int_0^t \mathbb{E}[\mathrm{e}^{i\theta(N^1_{t-s}+N^2_{t-s})}]\lambda\mathrm{e}^{-\lambda s}\,ds
\end{aligned}$$

where in the last line we have split N_t into $N^1_{t-s} + N^2_{t-s}$, the sum of the offspring of particle 1 and the offspring of particle 2 in the time remaining from s to t. Now make a u-substitution and use the fact that N^1_{t-s}, N^2_{t-s} are independent and identically distributed to N_{t-s} to see that,

$$\mathbb{E}[\mathrm{e}^{i\theta N_t}] = \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \int_0^t \mathbb{E}[\mathrm{e}^{i\theta N_u}]^2 \lambda \mathrm{e}^{-\lambda t}\mathrm{e}^{\lambda u}\, du$$

Rather than solve this integral equation, we are going to plug in our guess, $f(\mathrm{e}^{-\lambda t}, \theta)$, for $\mathbb{E}[\mathrm{e}^{i\theta N_t}]$ and verify that the equality holds. That is, we want to show that

$$f(\mathrm{e}^{-\lambda t}, \theta) = \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\int_0^t f(\mathrm{e}^{-\lambda u}, \theta)^2 \lambda \mathrm{e}^{\lambda u}\, du \tag{A.13}$$

The right hand side becomes

$$\begin{aligned} \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\int_0^t f(\mathrm{e}^{-\lambda u}, \theta)^2 \lambda \mathrm{e}^{\lambda u}\, du &= \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\int_0^t \frac{\mathrm{e}^{-2\lambda u}\mathrm{e}^{2i\theta}}{(1-(1-\mathrm{e}^{-\lambda u})\mathrm{e}^{i\theta})^2} \lambda \mathrm{e}^{\lambda u}\, du \\ &= \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\mathrm{e}^{2i\theta}\int_0^t \frac{\lambda \mathrm{e}^{-\lambda u}}{(1-(1-\mathrm{e}^{-\lambda u})\mathrm{e}^{i\theta})^2}\, du \end{aligned}$$

Let $y = 1-(1-\mathrm{e}^{-\lambda u})\mathrm{e}^{i\theta}$ and the integral becomes

$$\begin{aligned} &= \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\mathrm{e}^{2i\theta}\int_1^{1-(1-\mathrm{e}^{-\lambda t})\mathrm{e}^{i\theta}} \frac{-\mathrm{e}^{-i\theta}}{y^2}\, dy \\ &= \mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\mathrm{e}^{i\theta}\left(\frac{1}{1-(1-\mathrm{e}^{-\lambda t})\mathrm{e}^{i\theta}} - 1\right) \\ &= \frac{\mathrm{e}^{i\theta}\mathrm{e}^{-\lambda t}}{1-(1-\mathrm{e}^{-\lambda t})\mathrm{e}^{i\theta}} \\ &= f(\mathrm{e}^{-\lambda t}, \theta) \end{aligned}$$

as desired. Therefore,

$$\mathbb{E}[\mathrm{e}^{i\theta N_t}] = f(\mathrm{e}^{-\lambda t}, \theta) \tag{A.14}$$

so N_t is geometric with parameter $\mathrm{e}^{-\lambda t}$. $\square$

A.3 A Brownian Motion Tail Bound

When bounding the motion of the particles in a branching Brownian motion, the many-to-one lemma often leaves us with an expression involving a single Brownian motion. We use the following tail bound to look at the probability that a Brownian motion travels far in a fixed amount of time.

Theorem 38. *Let $B(t)$ be a Brownian motion with $B(0) = 0$. Then*

$$\mathbb{P}(|B(h)| \geq \delta) \leq \frac{\sqrt{2h}}{\delta\sqrt{\pi}} \tag{A.15}$$

Proof. This proof relies on a simple change to the integral that represents this probability. We multiply by x/δ, which is greater than 1 because the integral we are taking starts at δ.

$$\begin{aligned}
\mathbb{P}(|B(h)| \geq \delta) &= 2\int_\delta^\infty \frac{1}{\sqrt{2\pi h}} \mathrm{e}^{-x^2/2h}\,dx \\
&\leq \frac{2}{\sqrt{2\pi h}} \int_\delta^\infty \frac{x}{\delta} \mathrm{e}^{-x^2/2h}\,dx \\
&= \frac{2\sqrt{h}}{\delta\sqrt{2\pi}} \left(-e^{-x^2/2h}\right)\Big|_\delta^\infty \\
&= \frac{2\sqrt{h}}{\delta\sqrt{2\pi}} \mathrm{e}^{-\delta^2/2h}
\end{aligned}$$

Sometimes this is the upper bound we choose to use. Other times, we go a step farther and say that

$$\mathbb{P}(|B(h)| \geq \delta) \leq \frac{\sqrt{2h}}{\delta\sqrt{\pi}}$$

□

A.4 Relevant Harris Chain Results

In Chapter 3, we prove that a Markov chain $Z(t)$ is in fact a positive recurrent Harris chain. We refer to several theorems in Meyn and Tweedie [17], which we state below for convenience of reference, using the notation of [17].

Theorem 39 ([17] Theorem 1.2(a)). *If $Z(t)$ is Harris recurrent with invariant measure π then $Z(t)$ is positive Harris recurrent if and only if there exists a closed petite set C such that for some (and then any) $\delta > 0$,*

$$\sup_{x\in C} \mathbb{E}_x[\tau_C(\delta)] < \infty$$

where $\tau_C(\delta)$ is the first hitting time of C after δ.

Theorem 40 ([17] Theorem 3.1). *Suppose that a is a general probability distribution on $\mathbb{R}^+$ and let $\{T(k)\}$ be an undelayed renewal process with increment distribution a. Then the K_a-chain of $Z(t)$ is the discrete time chain $Z_k = Z(T(k))$. If Z_k is Harris recurrent, then so is the process $Z(t)$. And then the chain Z_k is positive Harris recurrent if and only if the process $Z(t)$ is positive Harris recurrent.*

When this result in used in this thesis, we choose a to be the distribution which is a constant $T = O(\ln(N))$ with probability 1. This is a natural choice, as the process can be easily thought about in chunks of order $\ln(N)$, the amount of time it takes a Brownian motion to grow to size N.

Theorem 41 ([17] Theorem 3.3). *If C is petite and $\mathbb{P}_x(\tau_C < \infty) = 1$ for all $x \in X$, then $Z(t)$ is Harris recurrent.*

A.5 Continuity Statements Relating to Chapter 5

Lemma 42. *If $\Phi(x)$ is Lipschitz and $\Lambda(x)$ is uniformly continuous, then $\lambda(x,\cdot)$ is uniformly continuous. That is, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $Wass_1(\mu,\nu) < \delta$, then $|\lambda(x,\mu) - \lambda(x,\nu)| < \varepsilon$.*

Proof. Fix an $\varepsilon > 0$ and let K be the Lipschitz constant of Φ. Choose δ such that if $|x - y| < K\delta$, then $|\Lambda(x) - \Lambda(y)| < \varepsilon$, which we can do uniformly for all x. Let

$Wass_1(\mu, \nu) < \delta$. Then note the following:

$$|\lambda(x,\mu) - \lambda(x,\nu)| = \left|\Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu(dy)\right) - \Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\nu(dy)\right)\right|$$

Notice that

$$\begin{aligned}\left|\int_{\mathbb{R}} \Phi(x-y)\mu(dy) - \int_{\mathbb{R}} \Phi(x-y)\nu(dy)\right| &= K\left|\int_{\mathbb{R}} \frac{\Phi(x-y)}{K}\mu(dy) - \int_{\mathbb{R}} \frac{\Phi(x-y)}{K}\nu(dy)\right| \\ &\leq K\delta\end{aligned} \tag{A.16}$$

because $\frac{\Phi(x-y)}{K} = f_x(y)$ is a 1-Lipschitz function and therefore $\left|\int_{\mathbb{R}} f_x(y)\mu(dy) - \int_{\mathbb{R}} f_x(y)\nu(dy)\right| < \delta$ by the definition of the Wasserstein distance. Because the arguments are within $K\delta$ of each other, we know that

$$\left|\Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu(dy)\right) - \Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\nu(dy)\right)\right| < \varepsilon$$

Therefore, $|\lambda(x,\mu) - \lambda(x,\nu)| < \varepsilon$ as desired. □

Lemma 43. *If $\Phi(x)$ is Lipschitz and $\Lambda(x)$ is uniformly continuous, then $\lambda(\cdot,\mu)$ is uniformly continuous. That is, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x-y| < \delta$, then $|\lambda(x,\mu) - \lambda(y,\mu)| < \varepsilon$.*

Proof. Fix an $\varepsilon > 0$ and again let K be the Lipschitz constant of Φ. Because Λ is uniformly continuous, we can pick a δ such that if $|u - v| < K\mu(\mathbb{R})\delta$, then

$|\Lambda(u) - \Lambda(v)| < \varepsilon$. Suppose $|x - y| < \delta$. Then

$$\begin{aligned}
\left|\int_{\mathbb{R}} \Phi(x-z)\mu(dz) - \int_{\mathbb{R}} \Phi(y-z)\mu(dz)\right| &= \left|\int_{\mathbb{R}} (\Phi(x-z) - \Phi(y-z))\mu(dz)\right| \\
&\leq \int_{\mathbb{R}} |\Phi(x-z) - \Phi(y-z)|\mu(dz) \\
&\leq \int_{\mathbb{R}} K|x-y|\mu(dz) \\
&= K|x-y|\mu(\mathbb{R}) \\
&\leq K\mu(\mathbb{R})\delta
\end{aligned}$$

Therefore, if $|x - y| < \delta$, then

$$\begin{aligned}
|\lambda(x,\mu) - \lambda(y,\mu)| &= \left|\Lambda\left(\int_{\mathbb{R}} \Phi(x-z)\mu(dz)\right) - \Lambda\left(\int_{\mathbb{R}} \Phi(y-z)\mu(dz)\right)\right| \\
&\leq \varepsilon
\end{aligned}$$

by the choice of δ. Therefore, we have the desired inequality. $\square$

The next lemma tells us that we can bound the change in λ over a fixed time interval by a constant times the change in particle positions plus a constant times the length of the interval. The second term in the bound is necessary because each new particle birth adds mass to the system and therefore causes a jump in the rate of the process. The previous lemmas tell us that λ is continuous; roughly this says that between birth rates, if none of the particles move too far, then the branch rate of each particle doesn't change very much.

Lemma 44. *Suppose that Λ, Φ are bounded Lipschitz functions with Lipschitz constants K_Λ, K_Φ respectively and $||\Phi||_\infty = M_\Phi$. Fix a time interval $[t, t+h]$. Let N_t be the number of particles at time t and $N_{\Delta h} = N_{t+h} - N_t$. For this lemma, we write $X_u(t) = X_u(t)\mathbb{1}_{\tau_u \leq t} + X_v(t)\mathbb{1}_{\tau_u > t}$, where $v < u$. If $\sup\limits_{u \in A_{t+h}} \sup\limits_{t \leq s \leq t+h} |X_u(s) - X_u(t)| < \delta$*

and $N_{\Delta h} \leq Ch$, then there exists constants $C_1, C_2 > 0$, functions of N, N_t, such that $\sup_{t \leq s \leq t+h} |\lambda(x, \mu_s^N) - \lambda(x, \mu_t^N)| \leq C_1\delta + C_2 h.$

Proof. Using the definition of λ, we can see that

$$\begin{aligned}
&\left|\Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu_s^N(dy)\right) - \Lambda\left(\int_{\mathbb{R}} \Phi(x-y)\mu_t^N(dy)\right)\right| \\
&\leq K_\Lambda \left|\int_{\mathbb{R}} \Phi(x-y)\mu_s^N(dy) - \int_{\mathbb{R}} \Phi(x-y)\mu_t^N(dy)\right| \\
&= \frac{K_\Lambda}{N}\left|\sum_{u \in A_s} \Phi(x - X_u(s)) - \sum_{u \in A_t} \Phi(x - X_u(t))\right| \\
&\leq \frac{K_\Lambda}{N}\left(\sum_{u \in A_t} |\Phi(x - X_u(s)) - \Phi(x - X_u(t))| + \sum_{u \in A_s \setminus A_t} |\Phi(x - X_u(s))|\right) \qquad \text{(A.17)} \\
&\leq \frac{K_\Lambda}{N}\left(N_t K_\Phi \delta + ChM_\Phi\right) \\
&= \frac{K_\Lambda K_\Phi N_t}{N}\delta + \frac{K_\Lambda M_\Phi C}{N} h
\end{aligned}$$

□

Lemma 45. *Suppose the conditions of the above lemma are satisfied. Then there exists constants $D_1, D_2 > 0$, functions of N, N_t, such that for each particle k alive at time t,* $\sup_{t \leq s \leq t+h} |\lambda(X_u(s), \mu_s^N) - \lambda(X_u(t), \mu_t^N)| \leq D_1\delta + D_2 h.$

Proof. We can split up the change in λ to account for the change resulting from a shift in the x value and the change resulting from a shift in the measure.

$$\begin{aligned}
&|\lambda(X_u(s), \mu_s^N) - \lambda(X_u(t), \mu_t^N)| \\
&< |\lambda(X_u(s), \mu_s^N) - \lambda(X_u(s), \mu_t^N)| + |\lambda(X_u(s), \mu_t^N) - \lambda(X_u(t), \mu_t^N)| \qquad \text{(A.18)} \\
&< C_1\delta + C_2 h + |\lambda(X_u(s), \mu_t^N) - \lambda(X_u(t), \mu_t^N)|
\end{aligned}$$

But we have seen already that $\lambda(\cdot, \mu)$ is continuous, so it will not be hard to bound this by something proportional to δ.

$$
\begin{aligned}
&|\lambda(X_u(s), \mu_t^N) - \lambda(X_u(t), \mu_t^N)| \\
\left|\Lambda\left(\int_{\mathbb{R}} \Phi(X_u(s) - z)\mu_t^N(dz)\right) - \Lambda\left(\int_{\mathbb{R}} \Phi(X_u(t) - z)\mu_t^N(dz)\right)\right| & \\
&\leq K_\Lambda \left|\int_{\mathbb{R}} (\Phi(X_u(s) - z) - \Phi(X_u(t) - z))\mu_t^N(dz)\right| \\
&\leq K_\Lambda \int_{\mathbb{R}} |\Phi(X_u(s) - z) - \Phi(X_u(t) - z)|\mu_t^N(dz) \\
&\leq K_\Phi K_\Lambda \int_{\mathbb{R}} |X_u(s) - X_u(t)|\mu_t^N(dz) \\
&= K_\Phi K_\Lambda \delta \frac{N_t}{N} \\
&\leq C_3\delta
\end{aligned}
\tag{A.19}
$$

Therefore, we have that

$$
\begin{aligned}
|\lambda(X_u(s), \mu_s^N) - \lambda(X_u(t), \mu_t^N)| &< C_1\delta + C_2 h + C_3\delta \\
&= D_1\delta + D_2 h
\end{aligned}
\tag{A.20}
$$

giving us the desired bound. □

A.6 Poisson Facts

Theorem 46. *Let N_t be a Poisson random variable with rate λ. Then $\frac{N_t}{t} \to \lambda$ a.s. and in L^1 as $t \to \infty$.*

Proof. We begin with the a.s. convergence. Letting $\lfloor t \rfloor$ be the integer part of t, we can write $N(t) = N(\lfloor t \rfloor) + (N(t) - N(\lfloor t \rfloor)) = \sum_{k=1}^{\lfloor t \rfloor} X_k + (N(t) - N(\lfloor t \rfloor))$, where the X_k are i.i.d. Poi(1) random variables. Because X_k is integrable, we can apply the strong law of large numbers to say that $\frac{1}{\lfloor t \rfloor} N(\lfloor t \rfloor) \to \lambda$ a.s. We then have the following

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{N(\lfloor t\rfloor)}{t}+\frac{(N(t)-N(\lfloor t\rfloor))}{t}$$

$$=\lim_{t\to\infty}\frac{N(\lfloor t\rfloor)}{\lfloor t\rfloor}\frac{\lfloor t\rfloor}{t}+\lim_{t\to\infty}\frac{N(t)-N(\lfloor t\rfloor)}{t}$$

$$=\lambda+\lim_{t\to\infty}\frac{N(t)-N(\lfloor t\rfloor)}{t}$$

Notice that $0\le\frac{N(t)-N(\lfloor t\rfloor)}{t}$ and

$$\frac{N(t)-N(\lfloor t\rfloor)}{t}\le\frac{N(\lfloor t\rfloor+1)-N(\lfloor t\rfloor)}{t}$$

$$\le\frac{N(\lfloor t\rfloor+1])-N(\lfloor t\rfloor)}{\lfloor t\rfloor}$$

$$=\frac{X_{\lfloor t\rfloor+1}}{\lfloor t\rfloor}$$

But we know that $\lim\limits_{n\to\infty}\dfrac{\sum_{k=1}^{n+1}X_k}{n+1}=\lambda$, so

$$\lambda=\lim_{n\to\infty}\frac{X_{n+1}}{n}\frac{n}{n+1}+\frac{n}{n+1}\frac{1}{n}\sum_{k=1}^{n}X_k$$

$$=\lim_{n\to\infty}\frac{X_{n+1}}{n}+\lambda$$

which means that $\lim_{n\to\infty}\frac{X_{n+1}}{n}=0$. This says that

$$\lim_{t\to\infty}\frac{N(t)}{t}=\lambda+\lim_{t\to\infty}\frac{N(t)-N(\lfloor t\rfloor)}{t}$$

$$=\lambda \text{ a.s.}$$

To see L^1 convergence, we use the generalized LDCT. We note that $\left|\frac{N(t)}{t}-\lambda\right|\le\frac{N(t)}{t}+\lambda$ for all t and that by the argument above, $\frac{N(t)}{t}+\lambda$ converges pointwise to 2λ.

Additionally, $\mathbb{E}\left[\frac{N(t)}{t}+\lambda\right]=\frac{1}{t}\mathbb{E}[N(t)]+\lambda=2\lambda$, so $\mathbb{E}\left[\frac{N(t)}{t}+\lambda\right]$ clearly converges to $\mathbb{E}[2\lambda]=2\lambda$. Therefore, all the conditions of the generalized LDCT are satisfied and we can say that

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}\left[\frac{N(t)}{t}-\lambda\right]&=\mathbb{E}\left[\lim_{t\to\infty}\left|\frac{N(t)}{t}-\lambda\right|\right]\\&=0\end{aligned}$$

by the a.s. argument above. Therefore, we have L^1 convergence as well. $\square$

Bibliography

[1] David Aldous. Stopping times and tightness. *Ann. Probab.*, 6(2):335–340, 1978.

[2] Erin Beckman, Emily Dinan, Rick Durrett, Ran Huo, and Matthew Junge. Asymptotic behavior of the brownian frog model. *Electron. J. Probab.*, 23(paper no. 104):19 pp., 2017.

[3] Erin Beckman, Natalie Frank, Yufeng Jiang, Matthew Junge, and Si Tang. The frog model on trees with drift. *arXiv preprint arXiv: 1808.03283*, 2018.

[4] Jean B́erard and Jean-Baptiste Goúere. Brunet-derrida behavior of branching-selection particle systems on the line. *Commun. Math. Phys.*, 298:323–342, 2010.

[5] Julien Berestycki, Éric Brunet, John W. Harris, Simon C. Harris, and Matthew I. Roberts. Growth rates of the population in a branching brownian motion with an inhomogeneous breeding potential. *Stoch. Process. Appl.*, 125:2096–2145, 2015.

[6] Nathanael Berestycki and Lee Zhuo Zhao. The shape of multidimensional brunet–derrida particle systems. *Ann. Appl. Probab.*, 28(2):651–687, 2018.

[7] Patrick Billingsley. *Convergence of Probability Measures*. Wiley, 2nd edition, 1999.

[8] Eric Brunet and Bernard Derrida. Shift in the velocity of a front due to a cutoff. *Physical Review E*, 1997.

[9] Krzysztof Burdzy, Robert Holyst, David Ingerman, and Peter March. Configurational transition in a fleming-viot-type model and probabilistic interpretation of laplacian eigenfunctions. *J. Phys. A: Math. Gen.*, 29:2633–2642, 1996.

[10] Irina Cristali, Vinit Ranjan, Jake Steinberg, Erin Beckman, Rick Durrett, Matthew Junge, and James Nolen. Block size in geometric(p)-biased permutations. *Electron. Commun. Probab.*, 23(paper no. 80):10 pp., 2018.

[11] Anna De Masi, Pablo A. Ferrari, Errico Presutti, and Nahuel Soprano-Loto. Hydrodynamics of the n-bbm process. *arXiv preprint arXiv:1707.00799*, 2017.

[12] Rick Durrett. *Probability.* Cambridge University Press, 2010.

[13] Rick Durrett and Daniel Remenik. Brunet-derrida particle systems, free boundary problems and wiener-hopf equations. *Ann. Probab.*, 39(6):2043–2078, 2011.

[14] Simon C. Harris and Matthew I. Roberts. The many-to-few lemma and multiple spines. *Annales de l'Institut Henri Poincare - Probabilities et Statistiques*, 2017.

[15] Y. Iwasa, F. Michor, and MA Nowak. Stochastic tunnels in evolutionary dynamics. *Genetics*, 166(3), 2004.

[16] A. Joffe and M. Metivier. Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Adv. Appl. Prob.*, 18:20–65, 1986.

[17] Sean Meyn and R L. Tweedie. Generalized resolvents and harris recurrence of markov processes. *Contemporary Mathematics*, 149:227–250, 01 1993.

[18] U. Obolski, Y. Ram, and L. Hadany. Key issues review: evolution on rugged adaptive landscapes. *Rep. Prog. Phys.*, 81(1), 2018.

[19] Karl Oelschläger. A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Annals of Probab.*, 12(2):458–479, 1984.

[20] Karl Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 69:279–322, 1985.

[21] Karl Oelschläger. On the derivation of reaction-diffusion equations as limit dynamics of systems of moderately interacting stochastic processes. *Probab. Th. Rel. Fields*, 82:565–586, 1989.

[22] Sylvie Roelly-Coppoletta. A criterion of convergence of measure-valued processes: application to measure branching processes. *Stochastics*, 17:43–65, 1986.

[23] Feng Yu, Alison Etheridge, and Charles Cuthbertson. Asymptotic behavior of the rate of adaptation. *Ann. Appl. Probab.*, 20(3):978–1004, 2010.

Biography

Erin Melissa Beckman received her B.S. with Highest Honors in Mathematics and Chemistry from the University of Texas at Austin in December 2012. In May 2019, she earned her Ph.D. in Mathematics from Duke University. Erin will continue on to a postdoc at McGill University beginning September 2019.

Erin was supported by a National Science Foundation Research Training Grant from 2013-2017.

Erin has also co-authored papers on random permutations [10], the Brownian frog model on $\mathbb{R}^d$ [2], and the frog model with drift on $\mathbb{T}_d$ [3].